Bloom Filters, Adaptivity, and the Dictionary Problem

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Abstract

An approximate membership query data structure (AMQ)—such as a Bloom, quotient, or cuckoo filter—maintains a compact, probabilistic representation of a set $\mathcal{S}$ of keys from a universe $\mathcal{U}$. It supports lookups and inserts. Some AMQs also support deletes. A query for $x \in \mathcal{S}$ returns PRESENT. A query for $x \notin \mathcal{S}$ returns PRESENT with a tunable ***false-positive probability*** ε, and otherwise returns ABSENT.

AMQs are widely used to speed up dictionaries that are stored remotely (e.g., on disk or across a network). The AMQ is stored locally (e.g., in memory). The remote dictionary is only accessed when the AMQ returns PRESENT. Thus, the primary performance metric of an AMQ is how often it returns ABSENT for negative queries.

Existing AMQs offer weak guarantees on the number of false positives in a sequence of queries. The false-positive probability ε holds only for a single query. It is easy for an adversary to drive an AMQ's false-positive rate towards 1 by simply repeating false positives.

This paper shows what it takes to get strong guarantees on the number of false positives. We say that an AMQ is ***adaptive*** if it guarantees a false-positive probability of ε for every query, *regardless of answers to previous queries*.

We establish upper and lower bounds for adaptive AMQs. Our lower bound shows that it is impossible to build a small adaptive AMQ, even when the AMQ is immediately told whenever a query is a false positive. On the other hand, we show that it is possible to maintain an AMQ that uses the same amount of local space as a non-adaptive AMQ (up to lower order terms), performs all queries and updates in constant time, and guarantees that each negative query to the dictionary accesses remote storage with probability ε, independent of the results of past queries. Thus, we show that adaptivity can be achieved effectively for free.

1 INTRODUCTION

An approximate membership query data structure (AMQ)—such as a Bloom [4, 6], quotient [3, 30], single hash [29], or cuckoo [17] filter—maintains a compact, probabilistic representation of a set $\mathcal{S}$ of keys from a universe $\mathcal{U}$. It supports lookups and inserts. Some AMQs also support deletes. A positive query for $x \in \mathcal{S}$

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returns PRESENT. A negative query for $x \notin \mathcal{S}$ returns PRESENT with a tunable ***false-positive probability*** ε, and otherwise returns ABSENT.

AMQs are used because they are small. An optimal AMQ can encode a set $\mathcal{S} \subseteq \mathcal{U}$, where $|\mathcal{S}| = n$ and $|\mathcal{U}| = u$, with a false-positive probability ε using $\Theta(n \log(1/\varepsilon))$ bits [9]. In contrast, an error-free representation of $\mathcal{S}$ takes $\Omega(n \log u)$ bits.

One of the main uses of AMQs is to speed up dictionaries [6, 11, 14, 16, 18, 33, 35]. Often, there is not enough local storage (e.g., RAM) to store the dictionary's internal state, $\mathbf{D}$. Thus, $\mathbf{D}$ must be maintained remotely (e.g., on-disk or across a network), and accesses to $\mathbf{D}$ are expensive. By maintaining a local AMQ for the set $\mathcal{S}$ of keys occurring in $\mathbf{D}$, the dictionary can avoid accessing $\mathbf{D}$ on most negative queries: if the AMQ says that a key is not in $\mathcal{S}$, then no query to $\mathbf{D}$ is necessary.

Thus, the primary performance metric of an AMQ is how well it enables a dictionary to avoid these expensive accesses to $\mathbf{D}$. The fewer false positives an AMQ returns on a sequence of queries, the more effective it is.

AMQ guarantees. Existing AMQs offer weak guarantees on the number of false positives they will return for a sequence of queries. The false-positive probability of ε holds only for a single query. It does not extend to multiple queries, because queries can be correlated. It is easy for an adversary to drive an AMQ's false-positive rate towards 1 by simply repeating false-positives.

Even when the adversary is oblivious, i.e., it selects n queries without regard to the results of previous queries, existing AMQs have weak guarantees. With probability ε, a random query is a false positive, and repeating it n times results in a false-positive rate of 1. Thus, even when the adversary is oblivious, existing AMQs can have $O(\varepsilon n)$ false positives in expectation but not with high probability. This distinction has implications: Mitzenmacher et al. [25] show that on network traces, existing AMQs are suboptimal because they do not adapt to false positives.

Adaptive AMQs. We define an ***adaptive AMQ*** to be an AMQ that returns PRESENT with probability at most ε for every negative query, *regardless of answers to previous queries*. For a dictionary using an adaptive AMQ, *any* sequence of n negative queries will result in $O(\varepsilon n)$ false positives, with high probability. This gives a strong bound on the number of (expensive) negative accesses that the dictionary will need to make to $\mathbf{D}$. This is true even if the queries are selected by an adaptive adversary.

Several attempts have been made to move towards adaptivity (and beyond oblivious adversaries). Naor and Yogev [27] considered an adaptive adversary that tries to increase the false-positive rate by discovering collisions in the AMQ's hash functions, but they explicitly forbade the adversary from repeating queries. Chazelle et al. [10] introduced bloomier filters, which can be updated to specify a white list, which are elements in $\mathcal{U} - \mathcal{S}$ on which the AMQ may not answer PRESENT. However, bloomier filters are space efficient only when the white list is specified in advance, which makes them unsuitable for adaptivity. Mitzenmacher et al. [25] proposed an elegant variant of the cuckoo filter that stores part of the AMQ locally and part of it remotely in order to try to achieve adaptivity. They empirically show that their data structure helps maintain a low false-positive rate against queries that have temporal correlation.

However, no existing AMQ is provably adaptive.

Feedback, local AMQs, and remote representations. When an AMQ is used to speed up a dictionary, the dictionary always detects which are the AMQ's false positives and which are the true positives. Thus, the dictionary can provide this feedback to the AMQ. This feedback is free because it does not require any additional accesses to $\mathbf{D}$ beyond what was used to answer the query.

In this paper we show that, even with this feedback, it is impossible to construct an adaptive AMQ that uses less than $\Omega(\min\{n \log\log u, n \log n\})$ bits of space; see Theorem 5. That is, even if an AMQ is told which are the true and false positives, adaptivity requires large space.

This lower bound would appear to kill the whole idea of adaptive AMQs, since one of the key ideas of

game ADAPTIVITY-GAME($\mathcal{A}, n, \varepsilon$)
$\mathcal{O} \leftarrow$ SETUP(n, ε)
$x' \leftarrow \mathcal{A}^{\mathcal{O}}(n, \varepsilon)$
$b \leftarrow \mathcal{O}$.LOOKUP(x')
return $(b =$ PRESENT$) \wedge (x' \notin \mathcal{O}.\mathcal{S})$

function SETUP(n,ε)
$\mathcal{O}.\rho \xleftarrow{\$} \{0,1\}^{\mathbb{N}}$
$\mathcal{O}.\mathcal{S} \leftarrow \emptyset$
$(\mathcal{O}.\mathbf{L}, \mathcal{O}.\mathbf{R}) \leftarrow$INIT$(n, \varepsilon, \mathcal{O}.\rho)$
return $\mathcal{O}$

method $\mathcal{O}$.LOOKUP(x)
$(\mathbf{L}, b) \leftarrow$LOOKUP$(\mathbf{L}, x, \rho)$
if $(b =$PRESENT$) \wedge (x \notin \mathcal{S})$ **then**
$(\mathbf{L}, \mathbf{R}) \leftarrow$ADAPT$((\mathbf{L}, \mathbf{R}), x, \rho)$
return b

method $\mathcal{O}$.INSERT(x)
if $|\mathcal{S}| < n \wedge x \notin \mathcal{S}$ **then**
$(\mathbf{L}, \mathbf{R}) \leftarrow$INSERT$((\mathbf{L}, \mathbf{R}), x, \rho)$
$\mathcal{S} \leftarrow \mathcal{S} \cup \{x\}$

method $\mathcal{O}$.DELETE(x)
if $x \in \mathcal{S}$ **then**
$(\mathbf{L}, \mathbf{R}) \leftarrow$DELETE$((\mathbf{L}, \mathbf{R}), x, \rho)$
$\mathcal{S} \leftarrow \mathcal{S} \setminus \{x\}$

Figure 1: Definition of the game between an adaptive AMQ and an adversary $\mathcal{A}$. The adversary gets n, ε, and oracular access to $\mathcal{O}$, which supports three operations: $\mathcal{O}$.LOOKUP, $\mathcal{O}$.INSERT, and $\mathcal{O}$.DELETE. The adversary wins if, after interacting with the oracle, it outputs an element x' that is a false positive of the AMQ. An AMQ is adaptive if there exists a constant $\varepsilon < 1$ such that no adversary wins with probability greater than ε.

an AMQ is to be small enough to fit in local storage. Remarkably, efficient adaptivity is still achievable.

The way around this impasse is to partition an AMQ's state into a small local state $\mathbf{L}$ and a larger remote state $\mathbf{R}$. The AMQ can still have good performance, provided it access the remote state infrequently.

We show how to make an adaptive AMQ that consumes no more local space than the best non-adaptive AMQ (and much less than a Bloom filter). We call this data structure a ***broom filter*** (because it cleans up its mistakes). The broom filter accesses $\mathbf{R}$ only when the AMQ receives feedback that it returned a false positive.

When used to filter accesses to a remote dictionary $\mathbf{D}$, the AMQ's accesses to $\mathbf{R}$ are "free"—i.e. they do not asymptotically increase the number of accesses to remote storage—-because the AMQ access $\mathbf{R}$ only when the dictionary accesses $\mathbf{D}$.

Our lower bound shows that partitioning is essential to creating a space-efficient adaptive AMQ. Indeed, the adaptive cuckoo filter of Mitzenmacher et al. [25] also partitions its state into local and remote components, but it does not have the strong theoretical adaptivity guarantees of the broom filter.

The local component, $\mathbf{L}$, of the broom filter is itself a non-adaptive AMQ plus $O(n)$ bits for adaptivity. The purpose of $\mathbf{R}$ is to provide a little more information to help $\mathbf{L}$ adapt.

Thus, we have a dual view of adaptivity that helps us interpret the upper and lower bounds. The local representation $\mathbf{L}$ is an AMQ in its own right. The remote representation $\mathbf{R}$ is an "oracle" that gives extra feedback to $\mathbf{L}$ whenever there is a false positive. Because $\mathbf{R}$ is simply an oracle, all the heavy lifting is in the design of $\mathbf{L}$. In the broom filter, $\mathbf{R}$ enables $\mathbf{L}$ to identify an element $y \in \mathcal{S}$ that triggered the false positive.

Putting these results together, we pinpoint how much information is needed for an adaptive AMQ to update its local information. The lower bound shows that simply learning if the query is a false positive is not sufficient. But if this local information is augmented with asymptotically free remote lookups, then adaptivity is achievable.

A note on optimality. The broom filter dominates existing AMQs in all regards. Its local state by itself is an optimal conventional AMQ: it uses optimal space up to lower-order terms, and supports queries and updates in constant time with high probability. Thus the remote state is only for adaptivity. For comparison, a Bloom filter has a lookup time of $O(\log \frac{1}{\varepsilon})$, the space is suboptimal, and the filter does not support deletes. More recent AMQs [3, 17, 29, 30] also fail to match the broom filter on one or more of these criteria, even leaving aside adaptivity. Thus, we show that adaptivity has no cost.

2 PRELIMINARIES

We begin by defining the operations that our AMQ supports. These operations specify when it can access its local and remote states, when it gets to update its states, how it receives feedback, and basic correctness requirements (i.e., no false negatives). We define performance constraints (i.e., false-positive rates) later.

Definition 1 (***AMQs***)**.** *An approximate membership query data structure (AMQ) consists of the following deterministic functions. Here ρ denotes the AMQ's private infinite random string, $\mathbf{L}$ and $\mathbf{R}$ denote its private local and remote state, respectively, $\mathcal{S}$ represents the set of items that have been inserted into the AMQ more recently than they have been deleted, n denotes the maximum allowed set size, and ε denotes the false-positive probability.*

- $\text{INIT}(n, \varepsilon, \rho) \longrightarrow (\mathbf{L}, \mathbf{R})$. INIT *creates an initial state* $(\mathbf{L}, \mathbf{R})$.
- $\text{LOOKUP}(\mathbf{L}, x, \rho) \longrightarrow (\mathbf{L}', b)$. *For* $x \in \mathcal{U}$, LOOKUP *returns a new local state* $\mathbf{L}'$ *and* $b \in \{\text{PRESENT}, \text{ABSENT}\}$. *If* $x \in \mathcal{S}$, *then* $b = \text{PRESENT}$ *(i.e., AMQs do not have false negatives).* LOOKUP *does not get access to* $\mathbf{R}$.
- $\text{INSERT}((\mathbf{L}, \mathbf{R}), x, \rho) \longrightarrow (\mathbf{L}', \mathbf{R}')$. *For* $|\mathcal{S}| < n$ *and* $x \in \mathcal{U} \setminus \mathcal{S}$, INSERT *returns a new state* $(\mathbf{L}', \mathbf{R}')$. INSERT *is not defined for* $x \in \mathcal{S}$. DELETE *is defined analogously.*
- $\text{ADAPT}((\mathbf{L}, \mathbf{R}), x, \rho) \longrightarrow (\mathbf{L}', \mathbf{R}')$. *For* $x \notin \mathcal{S}$ *such that* $\text{LOOKUP}(\mathbf{L}, x, \rho) = \text{PRESENT}$, ADAPT *returns a new state* $(\mathbf{L}', \mathbf{R}')$.

An AMQ is ***local*** if it never reads or writes $\mathbf{R}$; an AMQ is ***oblivious*** if ADAPT is the identity function on $(\mathbf{L}, \mathbf{R})$. Bloom filters, cuckoo filters, etc, are local oblivious AMQs.

False positives and adaptivity. We say that x is a ***false positive*** of AMQ state $(\mathbf{L}, \mathbf{R})$ if $x \notin \mathcal{S}$ but $\text{LOOKUP}(\mathbf{L}, x, \rho)$ returns PRESENT.

We define an AMQ's false-positive rate using the adversarial game in Figure 1. In this game, we give the adversary access to the AMQ via an oracle $\mathcal{O}$. The oracle keeps track of the set $\mathcal{S}$ being represented by the AMQ and ensures that the adversary respects the limits of the AMQ (i.e., never overloads the AMQ, inserts an item that is already in $\mathcal{S}$, or deletes an item that is not currently in $\mathcal{S}$). The adversary can submit queries and updates to the oracle, which applies them to the AMQ and calls ADAPT whenever LOOKUP returns a false positive. The adversary cannot inspect the internal state of the oracle. ADAPTIVITY-GAME outputs TRUE iff the adversary wins, i.e., if, after interacting with the oracle, $\mathcal{A}$ outputs a false positive x' of the final state of the AMQ.

The ***static false-positive rate*** is the probability, taken over the randomness of the AMQ, that a particular $x \in \mathcal{U} \setminus \mathcal{S}$ is a false positive of the AMQ. This is equivalent to the probability that an adversary that never gets to query the AMQ is able to output a false positive. We formalize this as follows. An adversary is a ***single-query adversary*** if it never invokes $\mathcal{O}$.LOOKUP. We call this "single-query" because there is still an invocation of $\mathcal{O}$.LOOKUP at the very end of the game, when ADAPTIVITY-GAME tests whether x' is a false positive.

Definition 2. *An AMQ supports* ***static false-positive rate*** ε *if for all* n *and all single-query adversaries* $\mathcal{A}$*,*

$$\Pr[\text{ADAPTIVITY-GAME}(\mathcal{A}, n, \varepsilon) = \text{TRUE}] \leq \varepsilon.$$

Definition 3. *An AMQ supports* ***sustained false-positive rate*** ε *if for all* n *and all adversaries* $\mathcal{A}$*,*

$$\Pr[\text{ADAPTIVITY-GAME}(\mathcal{A}, n, \varepsilon) = \text{TRUE}] \leq \varepsilon.$$

An AMQ is ***adaptive*** if there exists a constant $\varepsilon < 1$ such that the AMQ guarantees a sustained false-positive rate of at most ε.

The following lemma shows that, since an adaptive AMQ accesses its remote state rarely, it must use as much local space as a local AMQ.

Lemma 4. *Any adaptive AMQ must have a local representation* $\mathbf{L}$ *of size at least* $n \log(1/\varepsilon)$.

Proof. Consider an adaptive AMQ with a sustained false positive rate of ε. Consider the local state $\mathbf{L}'$ at the time when the adversary provides x'. By the definition of sustained-false positive rate, $\mathbf{L}'$ must have a static false positive rate of at most ε. Thus, by the Bloom-filter lower bound [9, 23], $\mathbf{L}'$ must have size at least $n \log(1/\varepsilon)$. □

Cost model. We measure AMQ performance in terms of the RAM operations on $\mathbf{L}$ and in terms of the number of updates and queries to the remote representation $\mathbf{R}$. We measure these three quantities (RAM operations, remote updates, and remote queries) separately.

We follow the standard practice of analyzing AMQ performance in terms of the AMQ's maximum capacity, n.We assume a word size $w = \Omega(\log u)$ in most of the paper. For simplicity of presentation, we assume that $u = \text{poly}(n)$ but our results generalize.

Hash functions. We assume that the adversary cannot find a never-queried-before element that is a false positive of the AMQ with probability greater than ε. Ideal hash functions have this property for arbitrary adversaries. If the adversary is polynomially bounded, one-way functions are sufficient to prevent them from generating new false positives [27].

3 RESULTS

We prove the following lower bound on the space required by an AMQ to maintain adaptivity.

Theorem 5. *Any adaptive AMQ storing a set of size* n *from a universe of size* $u > n^4$ *requires* $\Omega(\min\{n \log n, n \log \log u\})$ *bits of space whp to maintain any constant sustained false-positive rate* $\varepsilon < 1$.

Together, Definition 1, Theorem 5 and Lemma 4 suggest what an optimal adaptive AMQ should look like. Lemma 4 says that $\mathbf{L}$ must have at least $n \log(1/\varepsilon)$ bits. Theorem 5 implies that any adaptive AMQ with $\mathbf{L}$ near this lower bound must make remote accesses.

A consequence of Definition 1 is that AMQs access $\mathbf{R}$ only when the system is accessing $\mathbf{D}$, so, if an AMQ performs $O(1)$ updates of $\mathbf{R}$ for each update of $\mathbf{D}$ and $O(1)$ queries to $\mathbf{R}$ for each query to $\mathbf{D}$, then accesses to $\mathbf{R}$ are asymptotically free. Thus, our target is an AMQ that has approximately $n \log(1/\varepsilon)$ bits in $\mathbf{L}$ and performs $O(1)$ accesses to $\mathbf{R}$ per update and query.

Our upper bound result is such an adaptive AMQ:

Theorem 6. *There exists an adaptive AMQ—the broom filter—that, for any sustained false-positive rate* ε *and maximum capacity* n*, attains the following performance:*

- ***Constant local work:*** $O(1)$ *operations for inserts, deletes, and lookups w.h.p.*
- ***Near optimal local space:*** $(1 + o(1))n \log \frac{1}{\varepsilon} + O(n)$ *local space w.h.p.*[1]
- ***Asymptotically optimal remote accesses:*** $O(1)$ *updates to* $\mathbf{R}$ *for each delete to* $\mathbf{D}$*;* $O(1)$ *updates to* $\mathbf{R}$ *with probability at most* ε *for each insertion to* $\mathbf{D}$*;* $O(1)$ *updates to* $\mathbf{R}$ *for each false positive.*

[1] All logarithms in this paper are base 2 unless specified otherwise.

The local component of the broom filter is, itself, an AMQ with performance that strictly dominates the Bloom Filter, which requires $(\log e)n\log(1/\varepsilon)$ space and $O(\log(1/\varepsilon))$ update time [4], and matches (up to lower-order terms) or improves upon the performance of more efficient AMQs [3, 17, 29, 31].

Since $\mathbf{L}$ contains an AMQ, one way to interpret our results is that a small local AMQ cannot be adaptive if it is only informed of true positives versus false positives, but it can adapt if it is given a little more information. In the case of the broom filter, it is given the element of S causing a false positive, that is, the element in S that has a hash function collision with the query, as we see next.

4 Broom Filters: Defining Fingerprints

The broom filter is a single-hash-function AMQ [3, 17, 29], which means that it stores fingerprints for each element in $\mathcal{S}$. In this section, we begin our proof of Theorem 6 by describing what fingerprints we store and how they establish the sustained false-positive rate of broom filters. In Section 5, we show how to maintain the fingerprints space-efficiently and in $O(1)$ time.

4.1 *Fingerprints*

The broom filter has a hash function $h : \mathcal{U} \to \{0, \ldots n^c\}$ for some constant $c \geq 4$. Storing an entire hash takes $c\log n$ bits, which is too much space—we can only afford approximately $\log(1/\varepsilon)$ bits per element. Instead, for set $\mathcal{S} = \{y_1, y_2, \ldots, y_n\}$, the broom filter stores a set of ***fingerprints*** $\mathcal{P} = \{p(y_1), p(y_2), \ldots, p(y_n)\}$, where each $p(y_i)$ is a ***prefix*** of $h(y_i)$, denoted $p(y_i) \sqsubseteq h(y_i)$.

Queries. A query for x returns PRESENT iff there exists a $y \in \mathcal{S}$ such that $p(y) \sqsubseteq h(x)$. The first $\log n + \log(1/\varepsilon)$ bits of a fingerprint comprise the ***baseline fingerprint***, which is subdivided as in a quotient filter [3, 30]. In particular, the first $q = \log n$ bits comprise the ***quotient***, and the next $r = \log(1/\varepsilon)$ bits the ***remainder***. The remaining bits (if any) comprise the ***adaptivity bits***.

Using the parts of the fingerprint. The baseline fingerprint is long enough to guarantee that the false-positive rate is at most ε. We add adaptivity bits to fix false positives, in order to achieve a sustained false-positive rate of ε. Adaptivity bits are also added during insertions. We maintain the following invariant:

Invariant 7. *No fingerprint is a prefix of another.*

By this invariant, a query for x can match at most one $p(y) \in \mathcal{P}$. As we will see, we can fix a false positive by adding adaptivity bits to the single $p(y)$, for which $p(y) \sqsubseteq h(x)$. Thus, adding adaptivity bits during insertions reduces the number of adaptivity bits added during false positives, which will allow us to achieve $O(1)$ work and remote accesses for each operation.

Shortly we will give a somewhat subtler reason why adaptivity bits are added during insertions—in order to defeat deletion-based timing attacks on the sustained false-positive rate.

Maintaining the fingerprints. Here we describe what the broom filter does on a call to ADAPT. In this section we drop $(\mathbf{L}, \mathbf{R})$ and ρ from the notation for simplicity.

We define a subroutine of ADAPT which we call EXTEND$(x, \mathcal{P})$. This function is used to maintain Invariant 7 and to fix false positives.

Observe that on a query x there exists at most one y for which $p(y) \sqsubseteq h(x)$, by Invariant 7. If such a y exists, the EXTEND$(x, \mathcal{P})$ operation modifies the local representation by appending adaptivity bits to $p(y)$ until $p(y) \not\sqsubseteq h(x)$. (Otherwise, EXTEND$(x, \mathcal{P})$ does nothing.) Thus, EXTEND performs remote accesses to REVLOOKUP$_{\mathcal{P}}$, where REVLOOKUP$_{\mathcal{P}}(x)$ returns the (unique) $y \in \mathcal{S}$ such that $p(y) \sqsubseteq h(x)$. REVLOOKUP$_{\mathcal{P}}$ is a part of $\mathbf{R}$, and can be implemented using a dictionary.

We can define ADAPT(x) as follows:

- **Queries.** If a query x is a false positive, we call EXTEND$(x, \mathcal{P})$, after which x is no longer a false positive.

- **Insertions.** When inserting an element x into $\mathcal{S}$, we first check if Invariant 7 is violated, that is, if there exists a $y \in \mathcal{S}$ such that $p(y) \sqsubseteq h(x)$.[2] If so, we call EXTEND$(x, \mathcal{P})$, after which $p(y) \not\sqsubseteq h(x)$. Then we add the shortest prefix of $h(x)$ needed to maintain Invariant 7.

- **Deletions.** Deletions do not make calls to ADAPT. We defer the details of the deletion operation until after we discuss how to reclaim bits introduced by ADAPT. For now we note the naïve approach of deleting an element's fingerprint is insufficient to guarantee a sustained false-positive rate.

4.2 *Reclaiming Bits*

Each call to ADAPT adds bits, and so we need a mechanism to remove bits. An amortized way to reclaim bits is to rebuild the broom filter with a new hash function every $\Theta(n)$ calls to ADAPT.

This change from old to new hash function can be deamortized without losing a factor of 2 on the space. We keep two hash functions, h_a and h_b; any element y greater than frontier z is hashed according to h_a, otherwise, it is hashed according to h_b. At the beginning of a ***phase***, frontier $z = -\infty$ and all elements are hashed according to h_a. Each time we call ADAPT, we delete the smallest constant $c > 1$ elements in $\mathcal{S}$ greater than z and reinsert them according to h_b. (Finding these elements requires access to $\mathbf{R}$; again this can be efficiently implemented using standard data structures.) We then set z to be the value of the largest reinserted element. When z reaches the maximum element in $\mathcal{S}$, we begin a new phase by setting $h_a = h_b$, picking a new h_b, and resetting $z = -\infty$. We use this frontier method for deamortization so that we know which hash function to use for queries: lookups on $x \leq z$ use h_b and those on $x > z$ use h_a.

Observation 8. *A hash function times out after $O(n)$ calls to* ADAPT.

Because every call to ADAPT introduces an expected constant number of adaptivity bits, we obtain:

Lemma 9. *In any phase,* ADAPT *introduces $O(n)$ adaptivity bits into the broom filter with high probability.*

Proof. By Observation 8, for some constant c_1, there are $c_1 n$ false positives before the entire AMQ gets rehashed. Constant c_1 is determined by the number of elements that get rehashed per false positive, and so can be tuned.

Each time there is a false positive, there is a collision with exactly one element by Invariant 7. Given that there is a collision, the probability that it can be resolved by extending fingerprints by i bits is 2^{-i}. Whenever an element is rehashed, its adaptivity bits get thrown out. Thus, by Chernoff bounds, the number of adaptivity bits in the data structure at any time is $O(c_1 n)$ w.h.p. □

If we did not have deletions, then Observation 8 and Lemma 9 would be enough to prove a bound on total size of all fingerprints—because adaptivity bits are removed as their hash function times out. To support deletions we introduce adaptivity bits via a second mechanism. We will show that this second mechanism also introduces a total of $O(n)$ adaptivity bits per phase.

4.3 *Deletions and Adaptivity Bits*

It is tempting to support deletions simply by removing fingerprints from $\mathcal{P}$, but this does not work. To see why, observe that false positives are eliminated by adding adaptivity bits. Removing fingerprints destroys history and reintroduces false positives. This opens up the data structure to timing attacks by the adversary.

[2] This step and the following assume x does not already belong to $\mathcal{S}$. If it does, we don't need to do anything during insertions.

We describe one such timing attack to motivate our solution. The adversary finds a false positive x and an element $y \in \mathcal{S}$ that collides with x. (It finds y by deleting and reinserting random elements until x is once again a false positive.) The attack then consists of repeatedly looking up x, deleting y, then inserting y. This results in a false positive on every lookup until x or y's hash function changes.

Thus, the broom filter needs to remember the history for deleted elements, since they might be reinserted. Only once y's hash function has changed can y's history be forgotten. A profligate approach is to keep the fingerprints of deleted elements as "ghosts" until the hash function changes. Then, if the element is reinserted, the adaptivity bits are already there. Unfortunately, remembering deleted elements can blow up the space by a constant factor, which we cannot afford.

Instead, we remember the adaptivity bits and quotient from each deleted element's fingerprint—but we forget the remainder. Only once the hash function has changed do we forget everything. This can be accomplished by including deleted elements in the strategy described in Section 4.2. (with deletions, we increase the requirement on adaptivity bits reclaimed at once to $c > 2$).

Now when a new element x gets inserted, we check whether there exists a ghost that matches $h(x)$. If so, then we give x at least the adaptivity bits of the ghost, even if this is more than needed to satisfy Invariant 7. This scheme guarantees the following:

Property 10. *If x is a false positive because it collides with y, then it cannot collide with y again until x or y's hash function times out (even if y is deleted and reinserted).*

4.4 *Sustained False-Positive Rate*

We now establish the sustained false-positive rate of broom filters. We begin by introducing notation:

Definition 11. *Hashes $h(x)$ and $h(y)$ have a **soft collision** when they have the same quotient. They have a **hard collision** when they have the same quotient and remainder. Hash $h(x)$ and fingerprint $p(y)$ have a **full collision** if $p(y) \sqsubseteq h(x)$.*

The hash function is fixed in this section, so we refer to x and y themselves as having (say) a soft collision, with the understanding that it is their hashes that collide.

Lemma 12. *The probability that any query has a hard collision with any of n fingerprints is at most ε.*

Proof. The probability that any query collides with a single fingerprint is $2^{-(\log n + \log(1/\varepsilon))} = \varepsilon/n$. Applying the union bound, we obtain the lemma. □

Lemma 13. *The sustained false-positive rate of a broom filter is ε.*

Proof. We prove that on any query $x \notin S$, $\Pr[\exists y \in \mathcal{S} \mid x \text{ has a full collision with } y] \leq \varepsilon$, regardless of the previous history. Any previous query that is a negative or a true positive has no effect on the data structure. Furthermore, deletions do not increase the chance of any full collision, so we need only consider false positives and insertions, both of which induce rehashing.

We say that $x \in \mathcal{U}$ and $y \in \mathcal{S}$ are ***related at time t*** if (1) there exists $t' < t$ such that x was queried at time t' and y was in $\mathcal{S}$ at t', and (2) between t' and t, the hash functions for x and y did not change. Suppose x is queried at time t. Then, by Property 10, if x and y are related at time t, then $\Pr[x \text{ is a false positive at } t] = 0$. If x and y are not related at time t, then $\Pr[x \text{ has a full collision with } y] \leq \Pr[h(x) \text{ has a hard collision with } h(y)]$. Finally, by Lemma 12, $\Pr[x \text{ is a false positive at } t] \leq \varepsilon$. □

4.5 *Space Bounds for Adaptivity Bits*

We first prove that at any time there are $O(n)$ adaptivity bits. Then we bootstrap this claim to show a stronger property: there are $\Theta(\log n)$ fingerprints associated with $\Theta(\log n)$ contiguous quotients, and these fingerprints have a total of $O(\log n)$ adaptivity bits w.h.p. (thus they can be stored in $O(1)$ machine words).

For the purposes of our proofs, we partition adaptivity bits into two classes: ***extend bits***, which are added by calls to EXTEND, and ***copy bits***, which are added on insertion due to partial matches with formerly deleted items. As some bits may be both extend and copy bits, we partition adaptivity bits by defining all the adaptivity bits in a fingerprint to be of the same type as the last bit, breaking ties in favor of extend. If an item is deleted and then reinserted, its bits are of the same type as when it first got them. (So if an item that gets extend bits is deleted and reinserted with the same adaptivity bits, then it still has extend bits.)

Lemma 14. *At any time, there are $O(n)$ adaptivity bits in the broom filter with high probability.*

Proof. Lemma 9 bounds the number of extend bits. We still need to bound the number of copy bits. We do so using a straightforward application of Chernoff bounds.

The number of quotients that have at least k extend bits is $O(n/k)$. This is because the total number of extend bits is $O(n)$. Therefore, the probability that $h(x)$ accumulates k extend bits is $O(1/(k2^k)$. (This is the probability that $h(x)$ matches a quotient with k extend bits times the probability that those extend bits match.)

Thus, the expected number of copy bits from length-k strings is $O(n/2^k)$, for $1 \leq k \leq \Theta(\log n)$. By Chernoff, these bounds also hold w.h.p. for $k \leq (\log n)/\log\log n$; for $k > (\log n)/\log\log n$ Chernoff bounds give that there are $O(\log n)$ bits from length-k strings w.h.p. Thus, the total number of adaptivity bits is $O(n)$, w.h.p. □

Lemma 15. *There are $\Theta(\log n)$ fingerprints associated with a range of $\Theta(\log n)$ contiguous quotients, and these fingerprints have $O(\log n)$ total extend bits w.h.p.*

Proof. As long as there are $O(n)$ adaptivity bits and $\Theta(n)$ stored elements, then no matter how the adaptivity bits are distributed: the first time that x is queried or inserted with hash function h, ADAPT is called with probability $\Theta(\varepsilon)$. By Chernoff bounds, before the phase (see Section 4.2) ends, there are $O(n/\varepsilon)$ distinct elements not in $\mathcal{S}$ that are ever queried and $O(n/\varepsilon)$ distinct elements that are ever inserted into $\mathcal{S}$.

We can now calculate an upper bound on the number of adaptivity bits at any time t. Recall that at the very beginning of phase ℓ, there is a unique hash function h_ℓ that is in use, because $h_{\ell-1}$ has expired, and $h_{\ell+1}$ has not been used yet. Any extend adaptivity bits that are in the broom filter at time t in phase ℓ were generated as a result of collisions generated by $h_{\ell-1}$, h_ℓ, or $h_{\ell+1}$.

Now consider all elements that were ever inserted or queried any time during phase $\ell-1$, ℓ, or $\ell+1$ with $h_{\ell-1}$, h_ℓ, or $h_{\ell+1}$. If we took all these elements, and inserted them one at a time into $\mathcal{S}$, calling ADAPT to resolve any collisions, this scheme would at least generate all the extend adaptivity bits that are present at time t.

It thus suffices to show that even with this overestimate, the fingerprints associated with a range of $\Theta(\log n)$ contiguous quotients have a total of $O(\log n)$ extend bits w.h.p. Call the $\Theta(\log n)$ quotients under consideration the ***group***. Define $0/1$-random variable $X_i = 1$ iff element x_i lands in the group and induces a call to extend. Thus, $\Pr[X_i = 1] \leq O(\varepsilon \log n/n)$. There are $O(n/\varepsilon)$ elements inserted, deleted, and/or queried in these rounds. Thus, by Chernoff bounds, the number of elements that land in this quotient group is $O(\log n/\varepsilon)$, and at most $O(\log n)$ of them get adaptivity bits w.h.p.

We bound the number of bits needed to resolve the collisions. There are $O(\log n)$ elements that land in this group. We model this as a balls and bins game, where elements land in the same bin if they share the same quotient and remainder. Let random variable K_i represent the number of elements in the ith nonempty bin. The expected number of bits that get added until all collisions are resolved is $2\sum_{i=1}^{O(\log n)} \log(K_i)$.

By the convexity of the log function, $\sum_{i=1}^{O(\log n)} \log(K_i) = O(\log n)$, regardless of the distribution of the elements into bins.

To achieve concentration bounds on this result, we upper bound this process by a different process. Each time we add a bit, there is a probability of at least $1/2$ that it matches with at most half of the remaining strings. Thus, the number of adaptivity bits is stochastically dominated by the number of coin flips we need until we get $\Theta(\log n)$ heads, which is $\Theta(\log n)$ w.h.p. □

Lemma 16. *There are $\Theta(\log n)$ fingerprints associated with a range of $\Theta(\log n)$ contiguous quotients, and these fingerprints have $O(\log n)$ total adaptivity bits w.h.p.*

Proof. We established the bound on extend bits in Lemma 15; now we focus on copy bits.

Consider any time t when there are n elements in the broom filter, and consider any group of $\Theta(\log n)$ contiguous quotients. By Chernoff bounds, $\Theta(\log n)$ of these n elements have hashes that have a soft collision with one of these quotients w.h.p. By Lemma 14, there are a total of $O(\log n)$ extend bits in this range. We now show that there are also a total of $O(\log n)$ copy bits.

The scheme from Section 4.3 can be described in terms of balls and bins as follows. There are $\Theta(\log n)$ bins, one for each quotient. Each ***string*** of adaptivity bits belongs in a bin. Some bins can have multiple strings (but by standard balls-and-bins arguments, the fullest bin has $O(\log n/\log\log n)$ strings of adaptivity bits). When a new element x is inserted, it lands in the bin determined by $h(x)$. Then $p(x)$ inherits the adaptivity bits in the bin iff $h(x)$ matches those adaptivity bits. (This means that any given string of adaptivity bits started out as extend bits, even if it got copied many times as copy bits.)

We now bound the number of adaptivity bits by considering a variation that adds more bits than the scheme from Section 4.3. For each element inserted into a bin, we keep appending copy bits as long as there is a match with some string of adaptivity bits in the bin. Once there is a mismatch with every string, we stop. Thus, while the scheme from Section 4.3 adds copy bits only on *complete* matches, we allow *prefix* matches while still retaining good bounds.

We again overestimate the bounds by assuming that the adaptivity bits are adversarially (rather than randomly) divided into bit strings and that the bit strings are adversarially distributed among the bins.

Let random variable K_i denote the number of adaptivity bit strings in the bin where the ith element lands. The first claim that we want to make is the following:

Claim. $\Pr[K_i \geq X] < O(1/X)$.

Proof. This follows from Markov's inequality and Lemma 15. Since w.h.p., the total number of adaptivity bits is at most $O(\log n)$, the expected number of bits in a bin, and thus the expected number of strings, is $O(1)$.

We next show the following claim, one of the cornerstones of the proof.

Claim. $\sum_{i=1}^{\Theta(\log n)} \log(K_i) = O(\log n)$.

Proof. By the previous claim,

$$\Pr[K_i \geq X] \leq \Pr[\text{we flip a coin and get at least } \log(X) - O(1) \text{ tails before any head}].$$

Therefore, the probability that $\sum_{i=1}^{c\log n} \log(K_i) = d\log n$ is at most the probability that we flip a coin $d\log n$ times and get at most $c\log n$ heads. For a suitable choice of constants c and d, this is polynomially small.

Next we bound the total number of adaptivity bits that the elements inherit. Element x_i lands in a bin with K_i adaptivity bit strings. Each time a bit is added, with probability at least $1/2$, the number of adaptivity strings that still match with $h(x_i)$ decreases by half. Specifically, suppose that k adaptivity strings still match x_i. With probability at least $1/2$, after the next bit reveal, at most $\lfloor k/2 \rfloor$ still match. So after an expected $\leq 2\log(K_i)$ bits, no adaptivity bit strings still match x_i. Once again this game is modeled as flipping a coin until until we get $\Theta(\log n)$ heads, and by Chernoff, only $\Theta(\log n)$ are needed w.h.p. □

5 BROOM FILTERS: IMPLEMENTING FINGERPRINTS

In Section 4, we showed how to use fingerprints to achieve a sustained false-positive rate of ε. In this section we give space- and time-efficient implementations for the fingerprint operations that are specified in Section 4. We explain how we store and manipulate adaptivity bits (Section 5.1), quotients (Section 5.2), and remainders. We describe two variants of our data structure, because there are two ways to manage remainders, depending on whether $\log(1/\varepsilon) \leq 2 \log \log n$, the ***small-remainder case*** (Section 5.3), or $\log(1/\varepsilon) > 2 \log \log n$, the ***large-remainder case*** (Section 5.4).

Bit Manipulation within Machine Words. In Section A, we show how to implement a variety of primitives on machine words in $O(1)$ time using word-level parallelism. The upshot is that from now on, we may assume that the asymptotic complexity for any operation on the broom filter is simply the number of machine words that are touched during the operation.

In Section A, we show how to implement a variety of primitives on machine words in $O(1)$ time using word-level parallelism; see Lemma 29. The upshot is that from now on, we may assume that the asymptotic complexity for any operation on the broom filter is simply the number of machine words that are touched during the operation.

5.1 *Encoding Adaptivity Bits and Deletion Bits*

We store adaptivity bits separately from the rest of the fingerprint. By Lemma 16, all of the adaptivity bits in any range of $\Theta(\log n)$ quotients fit in a constant number of words. Thus, all of the searches and updates to (both copy and extend) adaptivity bits take $O(1)$ time.

5.2 *Encoding Quotients*

Quotients and remainders are stored succinctly in a scheme similar to quotient filters [3, 30]; we call this high-level scheme ***quotienting***.

Quotienting stores the baseline fingerprints succinctly in an array of $\Theta(n)$ slots, each consisting of r bits. Given a fingerprint with quotient a and remainder b, we would like to store b in position a of the array. This allows us to reconstruct the fingerprint based on b's location. So long as the number of slots is not much more than the number of stored quotients, this is an efficient representation. (In particular, we will have a sublinear number of extra slots in our data structure.)

The challenge is that multiple fingerprints may have the same quotient and thus contend for the same location. Linear probing is a standard technique for resolving collisions: slide an element forward in the array until it finds an empty slot. Linear probing does not immediately work, however, since the quotient is supposed to be reconstructed based on the location of a remainder. The quotient filter implements linear probing by maintaining a small number (between 2 and 3) of metadata bits per array slot which encode the target slot for a remainder even when it is shifted to a different slot.

The standard quotient filter does not achieve constant time operations, independent of ε. This is because when the remainder length $r = \log(1/\varepsilon) = \omega(1)$, and the fingerprint is stored in a set of $\Omega(\log n)$ contiguous slots, there can be $\omega(1)$ locations (words) where the target fingerprint could be. (This limitation holds even when the quotient filter is half empty, in which case it is not even space efficient enough for Theorem 6.)

Nonetheless, the quotient filter is a good starting point for the broom filter because it allows us to maintain a multiset of baseline fingerprints subject to insertions, deletions, and queries. In particular, some queries will have a hard collision with multiple elements.[3] We need to compare the adaptivity bits of

[3]This is the main challenge in achieving optimality with the single-hash function bloom filters of Pagh et al. [29] or the backyard hashing construction of Arbitman et al. [1]. Instead we used techniques that permit the same element to be explicitly duplicated multiple times.

the query to the adaptivity bits of each colliding element. The quotienting approach guarantees that these adaptivity bits are contiguous, allowing us to perform multiple comparisons simultaneously using word-level parallelism. In particular, Lemma 15 ensures that the adaptivity bits for $O(\log n)$ quotients fit into $O(1)$ machine words.

5.3 *Broom Filter Design for the Small-Remainder Case*

In this section we present a data structure for the case that $r = O(\log \log n)$.

High Level Setup. Our data structure consists of a primary and a secondary level. Each level is essentially a quotient filter; however, we slightly change the insert and delete operations for the primary level in order to ensure constant-time accesses.

As in a quotient filter, the primary level consists of $n(1+\alpha)$ slots, where each slot has a remainder of size $r = \log(1/\varepsilon) = O(\log \log n)$. Parameter α denotes the subconstant extra space we leave in our data structure; thus the primary level is a quotient filter as described in Section 5.2, with space parameterized by α (and with slightly modified inserts, queries, and deletes). We require $\alpha \geq \sqrt{(9r \log \log n)/\log n}$.

The secondary level consists of a quotient filter with $\Theta(n/\log n)$ slots with a different hash function h_2. Thus, an element x has two fingerprints $p_1(x)$ and $p_2(x)$. The internals of the two levels are maintained entirely independently: Invariant 7 is maintained separately for each level, and adaptivity bits do not carry over from the primary level to the secondary level.

How to Perform Inserts, Queries and Deletes. To insert $y \in \mathcal{S}$, we first try to store the fingerprint $p_1(y)$ in the primary level. This uses the technique described in Section 5.2: we want to store the remainder in the slot determined by the quotient. If the slot is empty, we store the remainder of $p_1(y)$ in that slot. Otherwise, we begin using linear probing to look for an empty slot, updating the metadata bits accordingly; see [3, 30].

However, unlike in previous quotienting-based data structures, we stop our probing for an empty slot early: the data structure only continues the linear probing over $O((\log n)/r)$ slots (and thus $O(1)$ words). If all of these slots are full, the item gets stored in the secondary level. In Lemma 18 we show that it finds an empty slot in $O(1)$ words in the secondary level w.h.p.

We always attempt to insert into the primary level first. In particular, even if x is deleted from the secondary level while reclaiming bits (Section 4.2), we still attempt to insert x into the primary level first.

Queries are similar to inserts—to query for y, we calculate $p_1(y)$ and search for it in the primary level for at most $O((\log n)/r)$ slots; if this fails we calculate $p_2(y)$ and search for it in the secondary level.

Lemma 17. *With high probability, $O(n/\log^2 n)$ elements are inserted into the secondary level.*

Proof. Partition the primary level into ***primary bins*** of $(1+\alpha)(\log n)/r$ consecutive slots. An element is inserted into the secondary level only if it is inserted into a sequence of $\Omega((\log n)/r)$ full slots; for this to happen either the primary bin containing the element is full or the bin adjacent to it is full. We bound the number of full primary bins.

In expectation, each bin is (initially) hashed to by $(\log n)/r$ elements. Thus, by Chernoff bounds, the probability that a given primary bin is hashed to by at least $(1+\alpha)(\log n)/r$ elements is at most $\exp(-(\alpha^2 \log n)/(3r)) \leq 1/\log^3 n$.

Thus, in expectation, $n/\log^3 n$ primary bins are full. Since these events are negatively correlated, we can use Chernoff bounds, and state that $O(n/\log^3 n)$ primary bins are full with high probability.

Each primary bin is hashed to by $O(\log n)$ elements in expectation (even fewer, in fact). Using Chernoff, each primary bin is hashed to by $O(\log n)$ elements w.h.p.

Putting the above together, even if all $O(\log n)$ elements hashed into any of the $O(n/\log^3 n)$ overflowing primary bins (or either adjacent bin) are inserted into the secondary level, we obtain the lemma. □

Lemma 18. *With high probability, all items in the secondary level are stored at most $O(\log n/r)$ slots away from their intended slot.*

Proof. Partition the secondary level into ***secondary bins*** of $\Theta(\log n/r)$ consecutive slots. Thus, there are $\Theta(nr/\log^2 n)$ secondary bins. The lemma can only be violated if one of these bins is full.

By Lemma 17, we are inserting $O(n/\log^2 n)$ elements into these bins. By classical balls and bins analysis, because there are more bins than balls, the secondary bin with the most balls has $O((\log n)/\log\log n) = O((\log n)/r)$ elements with high probability. Thus, no secondary bin ever fills up with high probability. □

Performance. The $O(1)$ lookup time follows by definition in the primary level, and by Lemma 18 in the secondary level. The total space of the primary level is $O((1+\alpha)n\log(1/\varepsilon)) + O(n)$, and the total space of the second level is $O((n\log(1/\varepsilon))/\log n)$. We guarantee adaptivity using the ADAPT function defined in Section 4, which makes $O(1)$ remote memory accesses per insert and false positive query.

5.4 *Broom Filter for Large Remainders*

In this section we present a data structure for the the large-remainder case, $\log(1/\varepsilon) > 2\log\log n$. Large remainders are harder to store efficiently since only a small number can fit in a machine word. E.g., we are no longer guaranteed to be able to store the remainders from all hard collisions in $O(1)$ words w.h.p.

However, large remainders also have advantages. We are very likely to be able to search using only a small portion of the remainder—a portion small enough that many can be packed into $O(1)$ words. In particular, we can "peel off" the first $2\log\log n$ bits of the remainder, filter out collisions just based on those bits, and we are left with few remaining potential collisions. We call these ***partial collisions***.

So we have an initial check for uniqueness, then a remaining check for the rest of the fingerprint. This allows us to adapt the small-remainder case to handle larger remainders without a slowdown in time.

Data structure description. As before, our data structure consists of two parts. We refer to them as the ***primary level*** and the ***backyard***. This notation emphasizes the structural difference between the two levels and the relationship with backyard hashing [1]. Unlike the small-remainder case, we use only a single hash function.

The primary level consists of two sets of slots: ***signature slots*** of size $2\log\log n$, and ***remainder slots*** of size $r - 2\log\log n$. As in Section 5.3, the number of remainder slots is $(1+\alpha)n$ and the number of signature slots is $(1+\alpha)n$, where $\alpha \geq \sqrt{18\log^2\log n/\log n}$. Because the appropriate slot is found while traversing the signature slots, we only need to store metadata bits for the signature slots; they can be omitted for the remainder slots. The signature slots are stored contiguously; thus $O(\log n/\log\log n)$ slots can be probed in $O(1)$ time.

Each item is stored in the same remainder slot as in the normal quotient filter (see Subsection 5.2). The signature slots mirror the remainder slots; however, only the first $2\log\log n$ bits of the remainder are stored, the rest are stored in the corresponding remainder slot.

The primary level. To insert an element y, we first try to insert $p(y)$ in the primary level. We find the signature slot corresponding to the quotient of $p(y)$. We then search through at most $O(\log n/\log\log n)$ signatures to find a partial collision (a matching signature) or an empty slot. We use metadata bits as usual—the metadata bits guarantee that we only search through signatures that have a soft collision with $p(y)$.

If there is a partial collision—a signature that matches the first $2\log\log n$ bits of the remainder of $p(y)$—we insert $p(y)$ into the backyard. If there is no empty slot, we insert $p(y)$ into the backyard. If we find an empty slot but do not find a partial collision, we insert $p(y)$ into the empty slot; this means that we insert the signature into the empty signature slot, and insert the full remainder of $p(y)$ into the corresponding remainder slot. We update the metadata bits of the signature slots as in [3, 30].

Querying for an element x proceeds similarly. In the primary level, we find the signature slot corresponding to the quotient of $p(x)$. We search through $O(\log n/\log\log n)$ slots for a matching signature. If we find a matching signature, we look in the corresponding remainder slot to see if we have a hard collision; if so we return PRESENT. If we do not find a matching signature, or if the corresponding remainder slot does not have a hard collision, we search for $p(x)$ in the back yard.

The back yard. The back yard is a compact hash table that can store $O(n/\log n)$ elements with $O(1)$ worst-case insert and delete time [1, 13]. When we store an element y in the back yard, we store its entire hash $h(y)$. Thus, w.h.p. there are no collisions in the back yard. Since the back yard has a capacity for $\Theta(n/\log n)$ elements, and each hash has size $\Theta(\log n)$, the back yard takes up $\Theta(n)$ bits, which is a lower-order term.

Lemma 19. *The number of elements stored in the back yard is $O(n/\log^2 n)$ with high probability.*

Proof. An element is stored in the backyard only if 1. it is in a sequence of $\Omega(\log n/\log\log n)$ full slots, or 2. it has a partial collision with some stored element.

The number of elements that are in a sequence of full slots is $O(n/\log^2 n)$ with high probability; this follows immediately from Lemma 17 with $r = 2\log\log n$.

A query element x has a partial collision with an element y if they have the first $\log n + 2\log\log n$ bits of their fingerprint in common. Thus, x and y collide with probability $1/(n\log^2 n)$; thus x has a partial collision with $1/\log^2 n$ stored elements in expecation. The lemma follows immediately from Chernoff bounds. □

Performance.

The back yard requires $O(n)$ total space, since each hash is of length $O(\log n)$. The primary level requires $(1+\alpha)nr$ space for all primary slots, plus $O(n)$ extra space for the adaptivity bits stored as in Subsection 5.1.

Inserts, deletes, and queries require $O(1)$ time. The search for partial collisions involves $O(\log n/\log\log n)$ signature slots, which fit in $O(1)$ words; these can be searched in constant time. We look at a single remainder slot, which takes $O(1)$ time. If needed, any back yard operation requires $O(1)$ time as well.

6 A LOWER BOUND ON ADAPTIVE AMQS

In this section, we show that an AMQ cannot maintain adaptivity along with space efficiency. More formally, we show that any adaptive AMQ must use $\Omega(\min\{n\log n, n\log\log u\})$ bits. This means that if an AMQ is adaptive and the size of $\mathbf{L}$ is $o(\min\{n\log n, n\log\log u\})$ bits, then it must access $\mathbf{R}$. The proof itself does not distinguish between bits stored in $\mathbf{L}$ or $\mathbf{R}$. For convenience, we show that the lower bound holds when all bits are stored in $\mathbf{L}$; this is equivalent to lower bounding the bits stored in $\mathbf{L}$ and $\mathbf{R}$.

Interestingly, a similar lower bound was studied in the context of Bloomier filters [10]. The Bloomier filter is an AMQ designed to solve the problem of storing n items for which it must return PRESENT, along with a whitelist of $\Theta(n)$ items for which it must return ABSENT. Other queries must have a static false-positive rate of ε. Chazelle et al. [10] give a lower bound on any data structure that updates this whitelist dynamically, showing that such a data structure must use $\Omega(n\log\log(u/n))$ space. Their lower bound implies that if the adversary gives an AMQ a dynamic white list of false positives that it needs to *permanently* fix, then it must use too much space. In this section, we generalize this bound to all adaptive AMQ strategies.

6.1 *Notation and Adversary Model*

We begin by further formalizing our notation and defining the adversary used in the lower bound. We fix n and ε and drop them from most notation. We use $\text{BUILD}(\mathcal{S}, \rho)$ to denote the state that results from calling $\text{INIT}(n, \varepsilon, \rho)$ followed by $\text{INSERT}(x, \rho)$ for each $x \in \mathcal{S}$ (in lexicographic order).

Adversary Model. The adversary does not have access to the AMQ's internal randomness ρ, or any internal state $\mathbf{L}$ of the AMQ. The adversary can only issue a query x to the AMQ and only learns the AMQ's output—PRESENT or ABSENT—to query x.

The goal of the adversary is to adaptively generate a sequence of $O(n)$ queries and force the AMQ to either use too much space or to fail to satisfy a sustained false-positive rate of ε.

Let $\varepsilon_0 = \max\{1/n^{1/4}, (\log^2 \log u)/\log u\}$. Our lower bound is $m = |\mathbf{L}| = \Omega(n \log 1/\varepsilon_0)$. Note that $\varepsilon_0 \leq \varepsilon$; otherwise the classic AMQ lower bound of $m \geq n \log 1/\varepsilon$ [9, 23] is sufficient to prove Theorem 5. One can think of ε_0 as a maximum bound on the effective false positive rate—how often the AMQ encounters elements that need fixing.

Attack Description. First, the adversary chooses a set $\mathcal{S}$ of size n uniformly at random from $\mathcal{U}$. Then, the attack proceeds in rounds. The adversary selects a set Q of size n uniformly at random from $\mathcal{U} - \mathcal{S}$. Starting from Q, in each round, he queries the elements that were false positives in the previous round. To simplify analysis, we assume that the adversary orders his queries in lexicographic order. Let FP_i be the set of queries that are false positives in round $i \geq 1$. The attack:

1. In the first round, the adversary queries each element of Q.
2. In round $i > 1$, if $|\text{FP}_{i-1}| > 0$, the adversary queries each element in FP_{i-1}; otherwise the attack ends.

Classifying False Positives. The crux of our proof is that some false positives are difficult to fix—in particular, these are the queries where an AMQ is unable to distinguish whether or not $x \in \mathcal{S}$ by looking at its state $\mathbf{L}$.[4] We call $y \in \mathcal{U} \setminus \mathcal{S}$ an ***absolute false positive*** of a state $\mathbf{L}$ and randomness ρ if there exists a set $\mathcal{S}'$ of size n and a sequence of queries $(x_1, \ldots, x_t)$ such that $y \in \mathcal{S}'$ and $\mathbf{L}$ is the state of the AMQ when queries $x_1, \ldots, x_t$ are performed on $\text{BUILD}(\mathcal{S}', \rho)$. We use $\text{AFP}(\mathbf{L}, \mathcal{S}, \rho)$ to denote the set of absolute false positives of state $\mathbf{L}$, randomness ρ, and true-positive set $\mathcal{S}$. We call $(\mathcal{S}', (x_1, \ldots, x_t))$ a ***witness*** to y.

We call $y \in \mathcal{U} \setminus \mathcal{S}$ an ***original absolute false positive*** of $\mathcal{S}$ and ρ if and only if $y \in \text{AFP}(\text{BUILD}(\mathcal{S}, \rho), \mathcal{S}, \rho)$. We denote the set of original absolute false positives $\text{OFP}(\mathcal{S}, \rho) = \text{AFP}(\text{BUILD}(\mathcal{S}, \rho), \mathcal{S}, \rho)$.

As the AMQ handles queries, it will need to fix some previous false positives. To fix a false positive, the AMQ must change its state so that it can safely answer ABSENT to it. For a state $\mathbf{L}$, we define the set of elements that are no longer false positives by the set $\text{FIX}(\mathbf{L}, \mathcal{S}, \rho) = \text{OFP}(\mathcal{S}, \rho) \setminus \text{AFP}(\mathbf{L}, \mathcal{S}, \rho)$. Note that all fixed false positives are original absolute false positives.

As an AMQ cannot have false negatives, it cannot fix an original absolute false positive y unless it learns that $y \notin \mathcal{S}$. This is formalized in the next two observations.

Observation 20. *For any randomness ρ, set $\mathcal{S}$, and state $\mathbf{L}$ of the AMQ, if a query $x \in \text{AFP}(\mathbf{L}, \mathcal{S}, \rho)$, then* $\text{LOOKUP}(\mathbf{L}, x, \rho)$ *must return* PRESENT.

Observation 21. *Let $\mathbf{L}_1$ be a state of the AMQ before a query x and $\mathbf{L}_2$ be the updated state after x (that is, after invoking* LOOKUP *and possibly* ADAPT*). Let y be an absolute false positive of $\mathbf{L}_1$ with witness S_y. Then if y is not an absolute false positive of $\mathbf{L}_2$, then $x \in S_y$.*

[4] This is as opposed to easy-to-fix queries where, e.g., the AMQ answers PRESENT randomly to confuse an adversary. For all previous AMQs we are aware of, all false positives are absolute false positives.

6.2 *Analysis*

We start with an overview of the lower bound.

First, we recall a known result (Claim 23) that a space-efficient AMQ must start with a large number of original absolute false positives for almost all $\mathcal{S}$. Given that an AMQ has a large number of original absolute false positives, an adversary can discover a fraction of them through randomly chosen queries Q (Lemma 24).

Next, we show that through adaptive queries, the adversary forces the AMQ to fix almost all of these discovered original absolute false positives, for most sets Q (Lemma 25 and Lemma 26).

The crux of the proof relies on Lemma 27, which says that the AMQ cannot fix too many *extra* original absolute false positives during the attack—thus, it needs a large number of distinct "fixed" sets to cover all the different sets of original absolute false positives that the adversary forces the AMQ to fix. This is where we use that the AMQ only receives a limited amount of feedback on each false positive—it cannot fix more false positives without risking some false negatives.

Finally, we bound lower bound the space used by the AMQ by observing that there is a 1-to-1 mapping from "fixed" sets of original absolute false positives to AMQ states. Thus, we can lower bound the number of AMQ states (and hence the space needed to represent them) by lower-bounding the number of sets of original absolute false positives the adversary can force the AMQ to fix.

Observation 22. *For a given randomness ρ and set $\mathcal{S}$ of size n, consider two fixed false positive sets* $\text{FIX}(\mathbf{L}_1, \mathcal{S}, \rho)$ *and* $\text{FIX}(\mathbf{L}_2, \mathcal{S}, \rho)$*. Then if* $\text{FIX}(\mathbf{L}_1, \mathcal{S}, \rho) \neq \text{FIX}(\mathbf{L}_2, \mathcal{S}, \rho)$*, then* $\mathbf{L}_1 \neq \mathbf{L}_2$*.*

Discovering original absolute false positives through random queries. While for some special sets $\mathcal{S}$ given in advance, an AMQ may be able to store $\mathcal{S}$ very accurately (with very few false positives), this is not true for most random sets $\mathcal{S}$ chosen from the universe by the adversary. We note the following claim from Naor and Yogev [27].

Claim 23 ([27, Claim 5.3])**.** *Given any randomness ρ of AMQ using space $m \leq n \log 1/\varepsilon_0 + 4n$ bits, for any set $\mathcal{S}$ of size n chosen uniformly at random from $\mathcal{U}$, we have:* $\Pr_{\mathcal{S}}\left[|\text{OFP}(\mathcal{S}, \rho)| \leq u\varepsilon_0\right] \leq 2^{-n}$.

For the remainder of this section, we fix a set $\mathcal{S}^* \subseteq \mathcal{U}$ of size n such that $|\text{OFP}(\mathcal{S}^*, \rho)| > u\varepsilon_0$.[5] Let $\mathcal{Q}$ be the set of all possible query sets Q the adversary can choose, that is, $\mathcal{Q} = \{Q \subseteq \mathcal{U} \setminus \mathcal{S}^* \mid |Q| = n\}$. (We do not include S^* in the notation of $\mathcal{Q}$ for simplicity.) The following lemma follows immediately from Chernoff bounds.

Lemma 24. *For a fixed randomness ρ of an AMQ of size $m \leq n \log 1/\varepsilon_0 + 4n$ and fixed set $\mathcal{S}^*$ such that* $|\text{OFP}(\mathcal{S}^*, \rho)| > u\varepsilon_0$*, we have* $\Pr_{Q \in \mathcal{Q}}\left[|Q \cap \text{OFP}(\mathcal{S}^*, \rho)| = \Omega(n\varepsilon_0)\right] \geq 1 - 1/\text{poly}(n)$.

Forcing the adaptive AMQ to fix large number of original absolute false positives. From the definition of sustained false-positive rate, the AMQ must fix an ε fraction of false positives in expectation in each round. If the expected number of false positives that the AMQ has to fix in each round is high, classic concentration bounds imply that the AMQ must fix close to this expected number with high probability in each round. This implies that there must be a round where the AMQ fixes a large number of original absolute false positives. The next lemma formalizes this intuition.

For a given Q, let $\Phi(Q, \mathcal{S}^*, \rho)$ be the maximal-sized set of query elements (out of Q) that the AMQ has fixed simultaneously in any state. For $1 \leq i \leq t$, let $\mathbf{L}_i$ be the state of the AMQ after query x_i. Then we let $\Phi(Q, \mathcal{S}^*, \rho) = \text{FIX}(\mathbf{L}_{t'}, \mathcal{S}^*, \rho)$ for the smallest t' such that $|\Phi(Q, \mathcal{S}^*, \rho)| \geq \text{FIX}(\mathbf{L}_{t''}, \mathcal{S}^*, \rho)$ for any t''.

[5] With probability $1/2^n$, the adversary gets unlucky and chooses a set $\mathcal{S}^*$ that does not satisfy this property, in which case he fails. This is okay, because we only need to show *existence* of a troublesome set $\mathcal{S}^*$—and we in fact show the stronger claim that most $\mathcal{S}^*$ suffice.

The following lemma shows that the AMQ must, at the beginning of some round in the first $O(n)$ queries by the adversary, fix $\widetilde{\Omega}(n\varepsilon_0)$ false positives.

Lemma 25. *Consider an AMQ of size $m \le n \log 1/\varepsilon_0 + 4n$. For any set Q satisfying $Q \cap \text{OFP}(\mathcal{S}^*, \rho) = \Omega(n\varepsilon_0)$, there exists a round $T(Q,\rho)$ and a state $\mathbf{L}_{T(Q,\rho)}$ at the beginning of round $T(Q,\rho)$ such that $|\text{FIX}(\mathbf{L}_{T(Q,\rho)}, \mathcal{S}^*, \rho)| = \Omega(n\varepsilon_0/\log_\varepsilon \varepsilon_0)$ w.h.p., that is,*

$$\Pr_\rho\Bigg[|\Phi(Q,\mathcal{S}^*,\rho)| = \Omega(n\varepsilon_0/\log_\varepsilon \varepsilon_0) \;\Bigg|\; |Q \cap \text{OFP}(\mathcal{S}^*,\rho)| = \Omega(n\varepsilon_0)\Bigg] \ge 1 - 1/\text{poly}(n).$$

Round $T(Q,\rho)$ is reached in at most $O(n)$ total queries.

Proof. We fix Q and set $T = T(Q,\rho)$.

Recall that FP_T denotes the set of queries that are false positives in round T, and let $T_f = \log_\varepsilon \varepsilon_0$. Since the AMQ has a sustained false-positive rate of ε, we have $|\text{FP}_1| = O(n\varepsilon)$. As $\varepsilon \ge \varepsilon_0 \ge 1/n^{1/4}$, by Chernoff bounds, we have $|\text{FP}_{T+1}| \le \varepsilon|\text{FP}_T|(1 + 1/\log n)$ with high probability for all $1 \le T \le T_f$.

Suppose there does not exist a round $T < T_f$ such that the lemma holds, that is, in each round $T < T_f$, $|\text{FIX}(\mathbf{L}_T, \mathcal{S}^*, \rho)| \le n\varepsilon_0/2\log_\varepsilon \varepsilon_0$, where $\mathbf{L}_T$ is the state of the AMQ at the beginning of round T. In round T_f, the AMQ is asked $|\text{FP}_{T_f-1}| \le (\varepsilon(1 + 1/\log n))^{T_f-2} n = O(n\varepsilon_0)$ queries. From our assumption, $|\text{OFP}(\mathcal{S}^*,\rho) \cap \text{FP}_{T_f-1}| \ge n\varepsilon_0(1 - 1/2\log_\varepsilon \varepsilon_0)^{T_f-1} = \Omega(n\varepsilon_0)$.

To maintain a sustained false-positive rate of ε, it must hold that $|\text{FP}_{T_f}| = O(n\varepsilon\varepsilon_0)$ with high probability. Thus, in round T_f the AMQ must answer ABSENT to $\Omega(n(1-\varepsilon)\varepsilon_0) = \Omega(n\varepsilon_0)$ original absolute false positives from the set $|(\text{OFP}(\mathcal{S}^*,\rho) \cap \text{FP}_{T_f-1}) \setminus \text{FP}_{T_f}|$. We denote this set of original absolute false positives queries that the AMQ says ABSENT to in round T_f as $\mathcal{A}_{T_f}$.

Let $\mathbf{L}_{T_f,x}$ denote the state of the AMQ in round T_f just before query x is made. Then by Observation 20, $x \in \text{FIX}(\mathbf{L}_{T_f,x}, \mathcal{S}^*, \rho)$ for any $x \in \mathcal{A}_{T_f}$. We now show that all $x \in \mathcal{A}_{T_f}$ must simultaneously be in the set of fixed false positives of the state $\mathbf{L}_{T_f}$ at the beginning of round T_f. Note that $x \in \text{OFP}(\mathcal{S}^*,\rho) \cap \text{FP}_{T_{f-1}}$ and all queries between query x in round T_{f-1} and query x in round T_f are distinct from x and were chosen independently from x in round 1. As there can be at most n queries in between query x in consecutive rounds, using Observation 21, the probability that there exists a state $\mathbf{L}_i$ between $\mathbf{L}_{T_{f-1},q}$ and $\mathbf{L}_{T_f,q}$ such that $x \notin \text{FIX}(\mathbf{L}_i, \mathcal{S}^*, \rho)$ is at most $n^2/u < 1/n^2$. Thus, with high probability, $x \in \text{FIX}(\mathbf{L}_{T_f}, \mathcal{S}^*, \rho)$ for any given $x \in \mathcal{A}_{T_f}$. That is, $\mathcal{A}_{T_f} \subseteq \text{FIX}(\mathbf{L}_{T_f}, \mathcal{S}^*, \rho)$, and thus, $|\text{FIX}(\mathbf{L}_{T_f}, \mathcal{S}^*, \rho)| = \Omega(n\varepsilon_0)$.

Furthermore, round T is reached in $n + \sum_{i=1}^{T} \text{FP}_i \le n(1-\varepsilon^T)/(1-\varepsilon) = O(n)$ queries. □

For simplicity, let $\varepsilon_0' = \varepsilon_0/\log_\varepsilon \varepsilon_0$. (This does not affect our final bounds.) The next lemma follows from Lemmas 24 and 25 and shows that (for most ρ), most query sets Q satisfy Lemma 25 with high probability.

Lemma 26. *Given an AMQ of size $m \le n \log 1/\varepsilon_0 + 4n$ and set $\mathcal{S}^*$ such that $|\text{OFP}(\mathcal{S}^*,\rho)| \ge u\varepsilon_0$, for all but a $1/\text{poly}(n)$ fraction of Q, there exists a round $T(Q,\rho)$ such that the AMQ is forced to fix $\Omega(n\varepsilon_0')$ original absolute false positive queries w.h.p. over ρ, that is,*

$$\Pr_\rho\left[\Pr_{Q\in\mathcal{Q}}\left[|\Phi(Q,\mathcal{S}^*,\rho)| = \Omega(n\varepsilon_0')\right] \ge 1 - \frac{1}{\text{poly}(n)}\right] \ge 1 - 1/\text{poly}(n).$$

Proof.

$$\begin{aligned}
&\Pr_{\rho}\left[\Pr_{Q\in\mathcal{Q}}\left[|\Phi(Q,\mathcal{S}^*,\rho)| = \Omega\left(n\varepsilon_0'\right)\right] \geq 1 - \frac{1}{\text{poly}(n)}\right] \\
&\quad\geq \Pr_{\rho}\left[\begin{aligned}&\Pr_{Q\in\mathcal{Q}}\left[|\Phi(Q,\mathcal{S}^*,\rho)| = \Omega\left(n\varepsilon_0'\right) \;\middle|\; |Q\cap \text{AFP}(\mathcal{S}^*,\rho)| = \Omega(n\varepsilon_0)\right]\cdot \\ &\qquad\qquad\cdot \Pr_{Q\in\mathcal{Q}}\left[|Q\cap \text{AFP}(\mathcal{S}^*,\rho)| = \Omega(n\varepsilon_0)\right] \geq 1 - \frac{1}{\text{poly}(n)}\end{aligned}\right] \\
&\quad\geq \Pr_{\rho}\left[\Pr_{Q\in\mathcal{Q}}\left[|\Phi(Q,\mathcal{S}^*,\rho)| = \Omega\left(n\varepsilon_0'\right) \;\middle|\; |Q\cap \text{AFP}(\mathcal{S}^*,\rho)| = \Omega(n\varepsilon_0)\right] \geq 1 - \frac{1}{\text{poly}(n)}\right] \\
&\quad\geq \left(1 - \frac{1}{\text{poly}(n)}\right)
\end{aligned}$$

The second step is from Lemma 24, and the final step is from Lemma 25. □

Small AMQs cannot fix too many original absolute false positives. Next, we show that for a randomly-chosen y, knowing $y \notin \mathcal{S}^*$ is unlikely to give information to the AMQ about which set $\mathcal{S}^*$ it is storing. In particular, an AMQ may try to rule out false positives that may be correlated to a query y. For example, an AMQ may (without asymptotic loss of space) partition the universe into pairs of elements such that if it learns one item in a pair is a false positive, it is guaranteed that the other is as well. With such a strategy, the AMQ can succinctly fix two false positives per query instead of one. Could a more intricate strategy allow the AMQ to fix more false positives—even enough to be an obstacle to our lower bound? We rule out this possibility in the following lemma. On a random query, the AMQ is very unlikely to fix a given false positive. We use this to make a w.h.p. statement about the number of fixed elements using Markov's inequality in our final proof.

Lemma 27. *Let $(x_1, \ldots, x_t)$ be a sequence of queries taken from a uniformly-sampled random set $Q \subset \mathcal{U}$ of size n, and let $\mathbf{L}'$ be the state of the AMQ after these queries. For an element $y \in \mathcal{U}$, the probability over Q that y is a fixed false positive after these queries is n^2/u. That is, $\Pr_{Q\in\mathcal{Q}}\left[y \in \text{FIX}(\mathbf{L}', \mathcal{S}^*, \rho)\right] \leq n^2/u$.*

Proof. If $y \in \text{FIX}(\mathbf{L}', \mathcal{S}^*)$, then by Observation 21, for any witness set S' of y, $S' \cap Q$ must be nonempty. Fix a single witness set S'. For a given $s' \in S'$ and $x' \in Q$, $\Pr(s' = x') = 1/u$. Taking the union bound over all n^2 such pairs (s', x') we achieve the lemma. □

Final lower bound. We prove the desired lower bound.

Theorem 5. *Any adaptive AMQ storing a set of size n from a universe of size $u > n^4$ requires $\Omega(\min\{n\log n, n\log\log u\})$ bits of space whp to maintain any constant sustained false-positive rate $\varepsilon < 1$.*

Proof. Assume by contradiction that $m \leq n\log 1/\varepsilon_0 + 4n$. Recall that $\varepsilon_0 = \max\{1/n^{1/4}, (\log^2\log u)/\log u\})$. Applying Observation 22 to the state $\mathbf{L}_{T(Q,\rho)}$ from Lemma 25 we obtain

$$2^m \geq \left|\{\text{FIX}(\mathbf{L}_{T(Q,\rho)}, \mathcal{S}^*, \rho) \mid Q \subset \mathcal{U}, |Q| = n\}\right|.$$

We lower bound how many distinct fixed sets the AMQ needs to store for a given ρ.

By definition, $\Phi(Q, \mathcal{S}^*, \rho) \subseteq \text{FIX}(\mathbf{L}_{T(Q,\rho)}, \mathcal{S}^*, \rho)$. But, the set of fixed elements cannot be much bigger than the set of fixed queries. Consider an arbitrary $x \in \mathcal{U}$. By Lemma 27, x is a fixed false positive with probability at most n^2/u. Thus, $\mathbf{E}_{Q\in\mathcal{Q}}\left[|\text{FIX}(\mathbf{L}_{T(Q,\rho)}, \mathcal{S}^*, \rho), \mathcal{S}^*, \rho)|\right] = n^2$. By Markov's inequality, $\Pr_{Q\in\mathcal{Q}}\left[|\text{FIX}(\mathbf{L}_{T(Q,\rho)}, \mathcal{S}^*, \rho)| = O(n^2)\right] = \Omega(1)$. Then there exists a set[6] $\mathcal{Q}^* = \left\{Q \subseteq \mathcal{U} \;\middle|\; |Q| = n,\right.$

[6] $\mathcal{Q}^*$ is a function of ρ and $\mathcal{S}$; we omit this to simplify notation.

FIX $(\mathbf{L}_{T(Q,\rho)}, \mathcal{S}^*, \rho) = O(n^2)\}$ such that $|\mathcal{Q}^*| = \Omega(|\mathcal{Q}|)$. Thus, $\left|\{\text{FIX}\left(\mathbf{L}_{T(Q,\rho)}, \mathcal{S}^*, \rho\right) \mid Q \in \mathcal{Q}\}\right| \geq |\{\Phi(Q, \mathcal{S}^*, \rho) | Q \in \mathcal{Q}^*, |\Phi(Q, \mathcal{S}^*, \rho)| = \Omega(n\varepsilon_0')\}| / \binom{O(n^2)}{\Omega(n\varepsilon_0')}$.

Now, we count the number of distinct $\Phi(Q, \mathcal{S}^*, \rho)$. Let $\mathcal{Z} = \{Z \subseteq \text{OFP}(\mathcal{S}^*, \rho) \mid |Z| = \Theta(n\varepsilon_0')\}$. We show that with high probability a randomly chosen set $Z \in \mathcal{Z}$ belongs to the set of $\Phi(Q, \mathcal{S}^*, \rho)$ for some Q (because we choose Q uniformly at random, this probabilistic argument lower bounds the number of such Z immediately). Recall that $|\text{OFP}(\mathcal{S}^*, \rho)| > u\varepsilon_0$.

$$
\begin{aligned}
&\Pr_{Z \in \mathcal{Z}}[\exists Q \in \mathcal{Q}^* \text{ with } Z \subseteq \Phi(Q, \mathcal{S}^*, \rho)] \\
&\quad \geq \Pr_{\substack{Z \in \mathcal{Z} \\ Q \in \mathcal{Q}^* \text{ with } Q \supset Z}} [Z \subseteq \Phi(Q, \mathcal{S}^*, \rho)\}] \qquad (1) \\
&\quad \geq \Pr_{\substack{Z \in \mathcal{Z} \\ Q \in \mathcal{Q}^* \\ Q \supset Z}} \left[Z \subseteq \Phi(Q, \mathcal{S}^*, \rho) \middle| |\Phi(Q, \mathcal{S}^*, \rho)| = \Omega(n\varepsilon_0') \right] \\
&\qquad \cdot \Pr_{Q \in \mathcal{Q}^*} \Big[|\Phi(Q, \mathcal{S}^*, \rho)| = \Omega(n\varepsilon_0') \Big| \\
&\qquad\qquad |Q \cap \text{OFP}(\mathcal{S}^*, \rho)| = \Omega(n\varepsilon_0) \Big] \cdot \Pr_{Q \in \mathcal{Q}^*} [|Q \cap \text{OFP}(\mathcal{S}^*, \rho)| = \Omega(n\varepsilon_0)] \qquad (2) \\
&\quad \geq \left(1 \Big/ \binom{n}{\Omega(n\varepsilon_0')}\right) \left(1 - \frac{1}{\text{poly}(n)}\right).
\end{aligned}
$$

For the first term in step (2), note that if the AMQ is forced to fix $\Omega(n\varepsilon_0')$ queries out of query set Q, then the probability that a randomly chosen Z corresponds to this forced query set is at most 1/(total number of possible forced subsets of Q). There can be at most $\binom{n}{\Omega(n\varepsilon_0')}$ such subsets. The second and third term in step (2) follow from Lemma 26 and Lemma 24 respectively—because $|\mathcal{Q}^*| = \Omega(|\mathcal{Q}|)$, a simple probabilistic argument shows that the probability over $Q \in \mathcal{Q}^*$ rather than $Q \in \mathcal{Q}$ retains the high probability bounds. We lower bound the total number of distinct forced sets (ignoring the lower-order $1 - 1/\text{poly}(n)$ term) as

$$
\left|\{\Phi(Q, \mathcal{S}^*, \rho) \mid Q \in \mathcal{Q}^*\}\right| \geq \binom{u\varepsilon_0}{\Omega(n\varepsilon_0')} \Big/ \binom{n}{\Omega(n\varepsilon_0')}.
$$

Putting it all togetherand removing the less-than-constant $1/\text{poly}(n)$ term,

$$
2^m \geq \binom{u\varepsilon_0}{\Omega(n\varepsilon_0')} \Big/ \binom{n}{\Omega(n\varepsilon_0')} \binom{O(n^2)}{\Omega(n\varepsilon_0')}.
$$

Taking logs and simplifying, (recall $(x/y)^y \leq \binom{x}{y} \leq (xe/y)^y$),

$$
m = \Omega\left(n\varepsilon_0' \left(\log \frac{u \log(1/\varepsilon_0)}{n} - \log \frac{n}{\varepsilon_0'}\right)\right).
$$

We have $\log(u \log(1/\varepsilon_0)/n) - \log(n/\varepsilon_0') = \Omega(\log u)$ because $u \gg n^2/\varepsilon_0'$. Because ε is a constant, $\varepsilon_0' = \Omega(\varepsilon_0 / \log(1/\varepsilon_0))$. Thus, $m = \Omega\left(\frac{n\varepsilon_0 \log u}{\log(1/\varepsilon_0)}\right)$.

Using the definition of ε_0, we have two cases.

1. If $1/n^{1/4} \geq (\log^2 \log u)/\log u$, then $m = \Omega(n \log n \log u / \log \log n)$.
2. If $1/n^{1/4} < (\log^2 \log u)/\log u$, then $m = \Omega(n \log \log u)$.

In case 1, we get a contradiction to the assumption that $m \leq n \log 1/\varepsilon_0 + 4n$. In case 2 if $m \leq n \log\log u + 4n$, then we obtain a bound of $\Omega(n \log\log u)$. □

Matching upper bound. We give an AMQ construction that shows that the above bound is tight.

Lemma 28. *There exists an adaptive AMQ that can handle $O(n)$ adaptive queries using $O(\min\{n \log n, n \log\log u\})$ bits of space w.h.p.*

Proof. If $\log n = O(\log\log u)$, we build a standard bloom filter with false-positive probability $\varepsilon = 1/n^c$ for $c > 1$. The adversary never queries an original absolute false positive w.h.p. (If he does, the AMQ can store it without affecting the w.h.p. space bound.)

Otherwise, we have $\log\log u = o(\log n)$, and thus $n/\log u = n^{\Omega(1)}$. We build a standard bloom filter with $\varepsilon = \log\log u/\log u$; this requires $O(n \log\log u)$ space.

In $O(n)$ queries there can be $O(n \log\log u/\log u)$ false positives ; from the assumption on the size of u this is $n^{\Omega(1)}$ so Chernoff bounds imply $O(n \log\log u/\log u)$ false positives w.h.p. We store all false positives in a whitelist. Each requires $O(\log u)$ bits. Thus, the space used is $O(n \log\log u)$. □

7 Additional Related Work

Bloom filters [4] are a ubiquitous data structure for approximate membership queries and have inspired the study of many other AMQs; for e.g., see [1, 3, 10, 29, 32]. Describing all Bloom filter variants is beyond the scope of this paper; see surveys [6, 33]. Here, we only mention a few well-known AMQs in Section 7.1 and several results closely related to adaptivity in Section 7.2.

7.1 *AMQ-iary*

Bloom Filters [4]. The standard Bloom filter, representing a set $\mathcal{S} \subseteq \mathcal{U}$, is composed of m bits, and uses k independent hash functions $h_1, h_2, \ldots, h_k$, where $h_i : \mathcal{U} \to \{1, \ldots, m\}$. To insert $x \in \mathcal{S}$, the bits $h_i(x)$ are set to 1 for $1 \leq 1 \leq k$. A query for x checks if the bits $h_i(x)$ are set to 1 for $1 \leq i \leq k$—if they are, it returns "present", and else it returns "absent". A Bloom filter does not support deletes. For a false-positive probability of $\varepsilon > 0$, it uses $m = (\log e)n \log(1/\varepsilon)$ bits and has an expected lookup time of $O(\log(1/\varepsilon))$.

Single-Hash-Function AMQs [9, 29]. Carter et al. [9] introduced the idea of an AMQ that uses a single hash function that maps each element in the universe to one of $\lceil n/\varepsilon \rceil$ elements, and then uses a compressed exact-membership tester. Pagh et al. [29] show how to build the first near-optimal single hash-function AMQ by applying a universal hash function $h : U \to \{0, 1, \ldots, \lceil n/\varepsilon \rceil\}$ to $\mathcal{S}$, storing the resulting values using $(1 + o(1))n \log \frac{1}{\varepsilon} + O(n)$ bits. Their construction achieves $O(1)$ amortized insert/delete bounds and has been deamortized, in the case of inserts, by the backyard hashing construction of Arbitman et al. [1].

Quotient Filters [3, 30]. The quotient filter (QF) is a practical variant of Pagh et al.'s single-hash function AMQ [29]. The QF serves as the basis for our adaptive AMQ and is described in Section 4.

Cuckoo Filter [17] The cuckoo filter is also a practical variant of Pagh et al.'s single-hash function AMQ [28]. However, the implementation is based on cuckoo hashing rather than linear probing. This difference leads to performance advantages for some parameter settings [17, 30].

It would be interesting to develop a provably adaptive variant of the cuckoo filter. Its structure makes it difficult to use our approach of maintaining adaptivity bits . On the other hand, analyzing the Markov chain resulting from the heuristic approach to cuckoo-filter adaptivity in [25] has its own challenges.

7.2 *Previous Work on Adaptive AMQs and Adaptive Adversaries*

Bloomier filters. Chazelle et al. [10]'s *Bloomier filters* generalize Bloom filters to avoid a predetermined list of undesirable false positives. Given a set S of size n and a whitelist W of size w, a Bloomier filter stores a function f that returns "present" if the query is in the S, "absent" if the query is not in $S \cup W$, and "is a false positive" if the query is in W. Bloomier filters use $O((n + w) \log 1/\varepsilon)$ bits. The set $S \cup W$ cannot be updated without blowing up the space used and thus their data structure is limited to a static whitelist.

Adversarial-resilient Bloom filters. Naor and Yogev [27] study Bloom filters in the context of a repeat-free adaptive adversary, which queries elements until it can find a never-before-queried element that has a false-positive probability greater than ε. (They do not consider the false-positive probability of repeated queries.) They show how to protect an AMQ from a repeat-free adaptive adversaries using cryptographically secure hash functions so that new queries are indistinguishable from uniformly selected queries [27]. Hiding data-structure hash functions has been studied beyond AMQs (e.g., see [5, 19, 20, 24, 26]).

Adaptive cuckoo filter. Recently, Mitzenmacher et al. [25] introduced the *adaptive cuckoo filter* which removes false positives after they have been queried. They do so by consulting the remote representation of the set stored in a hash table. Their data structure takes more space than a regular cuckoo filter but their experiments show that it performs better on real packet traces and when queries have a temporal correlation.

Other AMQs and false-positive optimization. Retouched Bloom filters [15] and *generalized Bloom gilter* [22] reduce the number of false positives by introducing false negatives. Variants of Bloom filter that choose the number of hash functions assigned to an element based on its frequency in some predetermined query distribution have also been studied [7, 34].

Data Structures and Adaptive Adversaries Nonadaptive data structures lead to significant problems in security-critical applications—researchers have described algorithmic complexity attacks against hash tables [8, 12], peacock and cuckoo hash tables [2], (unbalanced) binary search trees [12], and quicksort [21]. In these attacks, the adversary adaptively constructs a sequence of inputs and queries that causes the data structure to exhibit its worst-case performance.

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A Using Word-Level Parallelism

This section explains that we can store and maintain small strings (fingerprints or metadata) compactly within words, while retaining $O(1)$ lookup (e.g., prefix match), insert, and delete. Based on this section, we can just focus on where (i.e., in which word) data is stored, and we use lemma Lemma 29 as a black box and assume that we can do all manipulations within machine words in $O(1)$ time. (For simplicity, we assume that machine words have $\Theta(\log n)$ bits, since words cannot be shorter and it does not hurt if they are longer.)

Lemma 29. *Consider the following input.*

- *A "query" bit string q that fits within a $O(1)$ $O(\log n)$-bit machine words.*
- *"Target" strings $s_1, s_2, \ldots, s_\ell$ concatenated together so that $s = s_1 \circ s_2 \circ \cdots \circ s_\ell$ fits within $O(1)$ $O(\log n)$-bit machine words. The strings need not be sorted and need not have the same length.*
- *Metadata bits, e.g., a bit mask that indicates the starting bit location for each s_i.*

Then the following query and update operations take $O(1)$ time. (Query results are returned as bit maps.)

1. Prefix match query. *Given a range $[j, k]$, find all s_i such that $i \in [j, k]$ and s_i is a prefix of q.*
2. Prefix-length query. *More generally, given a range $[j, k]$, for each s_i such that $i \in [j, k]$, indicate the length of the prefix match between q and s_i.*
3. Insert. *Given an input string $\hat{s}$ and target rank i, insert $\hat{s}$ into s as the new string of rank i.*
4. Delete. *Given a target rank i, delete s_i from s.*
5. Splice, concatenate, and other string operations. *For example, given i, concatenate s_i with s_{i+1}. Given i and length x, remove up to x bits from the beginning of s_i.*
6. Parallel inserts, deletes, splices, concatenations, and queries. *Multiple insert, delete, query, splice, or concatenate operations can be supported in parallel. For example, remove x bits from the beginning of all strings, and indicate which strings still have bits.*

Proof. These operations can be supported using standard compact and succinct data-structures techniques, in particular, rank, select, mask, shift, as well as sublinear-sized lookup tables.

Insertion, deletion, concatenation, splicing, etc., are handled using simple selects, bit-shifts, masks, etc.

We first explain how to find a string s_i that is a prefix match of q for the special case that q has length at most $\log n/8$. We partition s into chunks such that each chunk has size $\log n/8$. Now some s_j are entirely contained within one chunk and some straddle a chunk boundary. Since only $O(1)$ strings can straddle a chunk boundary, we can search each of these strings serially.

In contrast, there may be many strings that are entirely contained within a chunk, and these we need to search in parallel. We can use lookup tables for these parallel searches. Since there are at most $2^{\log n/8} = n^{1/8}$ input choices for query string q and at most $n^{1/4}$ input choices for the concatenated target strings (metadata bits and s), a lookup table with precomputed responses to all possible queries still takes $o(n)$ bits.

The remaining operations are supported similarly using rank, select, and lookup tables. For example, the more general case of querying larger q is also supported similarly by dividing q into chunks, comparing the chunks iteratively, and doing further parallel manipulation on s, also using lookup tables. In particular, since we compare q in chunks, we have to remove all strings s_i that already do not have a prefix in common with q as well as those shorter strings where no prefix is left. □