Bisimulations Meet PCTL Equivalences for Probabilistic Automata*

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Abstract. Probabilistic automata (PA) [20] have been successfully applied in the formal verification of concurrent and stochastic systems. Efficient model checking algorithms have been studied, where the most often used logics for expressing properties are based on PCTL [11] and its extension PCTL* [4]. Various behavioral equivalences are proposed for PAs, as a powerful tool for abstraction and compositional minimization for PAs. Unfortunately, the behavioral equivalences are well-known to be strictly stronger than the logical equivalences induced by PCTL or PCTL*. This paper introduces novel notions of strong bisimulation relations, which characterizes PCTL and PCTL* exactly. We also extend weak bisimulations characterizing PCTL and PCTL* without next operator, respectively. Thus, our paper bridges the gap between logical and behavioral equivalences in this setting.

1 Introduction

Probabilistic automata (PA) [20] have been successfully applied in the formal verification of concurrent and stochastic systems. Efficient model checking algorithms have been studied, where properties are mostly expressed in the logic PCTL, introduced in [11] for Markov chains, and later extended in [4] for Markov decision processes, where PCTL is also extended to PCTL*.

To combat the infamous state space problem in model checking, various behavioral equivalences, including strong and weak bisimulations, are proposed for PAs. Indeed, they turn out to be a powerful tool for abstraction for PAs, since bisimilar states implies that they satisfy exactly the same PCTL formulae. Thus, bisimilar states can be grouped together, allowing one to construct smaller quotient automata before analyzing the model. Moreover, the nice compositional theory for PAs is exploited for compositional minimization [5], namely minimizing the automata before composing the components together.

For Markov chains, i.e., PAs without nondeterministic choices, the logical equivalence implies also bisimilarity, as shown in [3]. Unfortunately, it does not hold in general, namely PCTL equivalence is strictly coarser than bisimulation – and their extension probabilistic bisimulation – for PAs. Even there is such a gap between behavior and logical equivalences, bisimulation based minimization is extensively studied in the literatures to leverage the state space explosion, for instance see [6, 1, 15].

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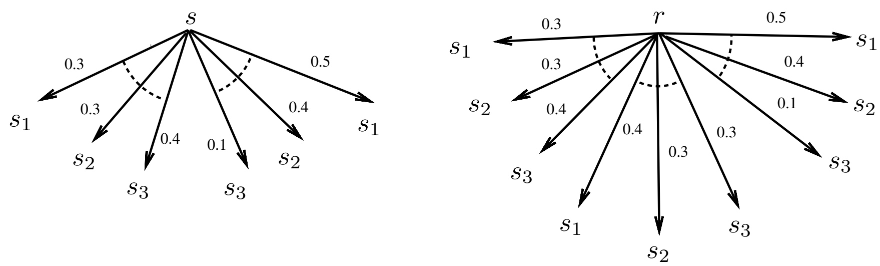

Fig. 1. Counter Example of Strong Probabilistic Bisimulation

The main reason for the gap can be illustrated by the following example. Consider the PAs in Fig.1 where assuming that s_1, s_2, s_3 are three absorbing states with different state properties. It is easy to see that s and r are PCTL equivalent: the additional middle transition out of r does not change the extreme probabilities. Existing bisimulations differentiate s and r, mainly because the middle transition out of r cannot be matched by any transition (or combined transition) of s. Bisimulation requires that the complete distribution of a transition must be matched, which is in this case too strong, as it differentiates states satisfying the same PCTL formulae.

In this paper we will bridge this gap. We introduce novel notions of behavioral equivalences which characterize (both soundly and completely) PCTL, PCTL* and their sublogics. Summarizing, our contributions are:

- A new bisimulation characterizing PCTL* soundly and completely. The bisimulation arises from a converging sequence of equivalence relations, each of which characterizes bounded PCTL*.
- Branching bisimulations which correspond to PCTL and bounded PCTL equivalences.
- We then extend our definitions to weak bisimulations, which characterize sublogics of PCTL and PCTL* with only unbounded path formulae.

Organization of the paper. Section 2 introduces some notations. In Section 3 we recall definitions of probabilistic automata, bisimulation relations by Segala [19]. We also recall the logic PCTL* and its sublogics. Section 4 introduces the novel strong and strong branching bisimulations, and proves that they agree with PCTL* and PCTL equivalences, respectively. Section 5 extends them to weak (branching) bisimulations. In Section 6 we discuss related work, and Section 7 concludes the paper.

2 Preliminaries

Probability space. A (discrete) probability space is a tuple $\mathcal{P} = (\Omega, F, \eta)$ where Ω is a countable set, $F = 2^{\Omega}$ is the power set, and $\eta : F \to [0,1]$ is a probability function which is countable additive. We skip F whenever convenient. Given probability spaces $\{\mathcal{P}_i = (\Omega_i, \eta_i)\}_{i \in I}$ and weights $w_i > 0$ for each i such that $\sum_{i \in I} w_i = 1$, the *convex combination* $\sum_{i \in I} w_i \mathcal{P}_i$ is defined as the probability space (Ω, η) such that $\Omega = \bigcup_{i \in I} \Omega_i$ and for each set $Y \subseteq \Omega$, $\eta(Y) = \sum_{i \in I} w_i \eta_i(Y \cap \Omega_i)$.

Distributions. We denote by $Dist(S)$ the set of discrete probability spaces over S. We shall use $s, r, t, \ldots$ and $\mu, \nu \ldots$ to range over S and $Dist(S)$, respectively. The support of μ is defined by $supp(\mu) := \{s \in S \mid \mu(s) > 0\}$. For an equivalence relation $\mathcal{R}$, we write $\mu \mathrel{\mathcal{R}} \nu$ if it holds that $\mu(C) = \nu(C)$ for all equivalence classes $C \in S/\mathcal{R}$. A distribution μ is called *Dirac* if $|supp(\mu)| = 1$, and we let $\mathcal{D}_s$ denote the Dirac distribution with $\mathcal{D}_s(s) = 1$.

Upward Closure. Below we define the upward closure of a subset of states.

Definition 1. *For pre-order $\mathcal{R}$ over S and $C \subseteq S$, define $\overline{C_{\mathcal{R}}} = \{s' \mid s \mathrel{\mathcal{R}} s' \wedge s \in C\}$. We say C is* $\mathcal{R}$-upward-closed *iff $C = \overline{C_{\mathcal{R}}}$.*

We use $\overline{s_{\mathcal{R}}}$ as the shorthand of $\overline{\{s\}_{\mathcal{R}}}$, and $\overline{\mathcal{R}} = \{\overline{C_{\mathcal{R}}} \mid C \subseteq S\}$ denotes the set of all $\mathcal{R}$-upward-closed sets.

3 Probabilistic Automaton, PCTL* and Bisimulations

Definition 2. *A* probabilistic automaton[3] *is a tuple $\mathcal{P} = (S, \rightarrow, IS, AP, L)$ where S is a finite set of states, $\rightarrow \ \subseteq S \times Dist(S)$ is a transition relation, $IS \subseteq S$ is a set of initial states, AP is a set of atomic propositions, and $L : S \rightarrow 2^{AP}$ is a labeling function.*

As usual we only consider image-finite PAs, i.e. $\{(r, \mu) \in \rightarrow \mid r = s\}$ is finite for each $s \in S$. A transition $(s, \mu) \in \rightarrow$ is denoted by $s \rightarrow \mu$. Moreover, we write $\mu \rightarrow \mu'$ iff for each $s \in supp(\mu)$ there exists $s \rightarrow \mu_s$ such that $\mu'(r) = \sum_{s \in supp(\mu)} \mu(s) \cdot \mu_s(r)$.

A *path* is a finite or infinite sequence $\omega = s_0 s_1 s_2 \ldots$ of states. For each $i \geq 0$ there exists a distribution μ such that $s_i \rightarrow \mu$ and $\mu(s_{i+1}) > 0$. We use $lstate(\omega)$ and $l(\omega)$ to denote the last state of ω and the length of ω respectively if ω is finite. The sets $Path$ is the set of all paths, and $Path(s_0)$ are those starting from s_0. Similarly, $Path^*$ is the set of finite paths, and $Path^*(s_0)$ are those starting from s_0. Also we use $\omega[i]$ to denote the $(i+1)$-th state for $i \geq 0$, $\omega|^i$ to denote the fragment of ω ending at $\omega[i]$, and $\omega|_i$ to denote the fragment of ω starting from $\omega[i]$.

We introduce the definition of *scheduler* to resolve nondeterminism. A scheduler is a function $\sigma : Path^* \rightarrow Dist(\rightarrow)$ such that $\sigma(\omega)(s, \mu) > 0$ implies $s = lstate(\omega)$. A scheduler σ is *deterministic* if it returns only Dirac distributions, that is, the next step is chosen deterministically. We use

$$Path(s_0, \sigma) = \{\omega \in Path(s_0) \mid \forall i \geq 0. \exists \mu. \sigma(\omega|^i)(\omega[i], \mu) > 0 \wedge \mu(\omega[i+1]) > 0\}$$

[3] In this paper we omit the set of actions, since they do not appear in the logic PCTL we shall consider later. Note that the bisimulation we shall introduce later can be extended to PA with actions directly.

to denote the set of paths starting from s_0 respecting σ. Similarly, $Path^*(s_0, \sigma)$ only contains finite paths.

The *cone* of a finite path ω, denoted by C_ω, is the set of paths having ω as their prefix, i.e., $C_\omega = \{\omega' \mid \omega \leq \omega'\}$ where $\omega' \leq \omega$ iff ω' is a prefix of ω. Fixing a starting state s_0 and a scheduler σ, the measure $Prob_{\sigma,s_0}$ of a cone C_ω, where $\omega = s_0 s_1 \ldots s_k$, is defined inductively as follows: $Prob_{\sigma,s_0}(C_\omega)$ equals 1 if $k = 0$, and for $k > 0$,

$$Prob_{\sigma,s_0}(C_\omega) = Prob_{\sigma,s_0}(C_{\omega|^{k-1}}) \cdot \left(\sum_{(s_{k-1},\mu') \in \rightarrow} \sigma(\omega|^{k-1})(s_{k-1}, \mu') \cdot \mu'(s_k) \right)$$

Let $\mathcal{B}$ be the smallest algebra that contains all the cones and is closed under complement and countable unions. [4] $Prob_{\sigma,s_0}$ can be extended to a unique measure on $\mathcal{B}$.

Given a pre-order $\mathcal{R}$ over S, $\overline{\mathcal{R}}^i$ is the set of $\mathcal{R}$-*upward-closed paths* of length i composed of $\mathcal{R}$-upward-closed sets, and is equal to the *Cartesian* product of $\overline{\mathcal{R}}$ with itself i times. Let $\overline{\mathcal{R}}^* = \cup_{i \geq 1} \overline{\mathcal{R}}^i$ be the set of $\mathcal{R}$-*upward-closed paths* of arbitrary length. Define $l(\Omega) = n$ for $\Omega \in \overline{\mathcal{R}}^n$. For $\Omega = C_0 C_1 \ldots C_n \in \overline{\mathcal{R}}^*$, the $\mathcal{R}$-*upward-closed cone* C_Ω is defined as $C_\Omega = \{C_\omega \mid \omega \in \Omega\}$, where $\omega \in \Omega$ iff $\omega[i] \in C_i$ for $0 \leq i \leq n$.

For distributions μ_1 and μ_2, we define $\mu_1 \times \mu_2$ by $(\mu_1 \times \mu_2)((s_1, s_2)) = \mu_1(s_1) \times \mu_2(s_2)$. Following [2] we also define the interleaving of PAs:

Definition 3. *Let $\mathcal{P}_i = (S_i, \rightarrow_i, IS_i, AP_i, L_i)$ be two PAs with $i = 1, 2$. The* interleave composition $\mathcal{P}_1 \,||\, \mathcal{P}_2$ *is defined by:*

$$\mathcal{P}_1 \,||\, \mathcal{P}_2 = (S_1 \times S_2, \rightarrow, IS_1 \times IS_2, AP_1 \cup AP_2, L)$$

where $L((s_1, s_2)) = L_1(s_1) \cup L_2(s_2)$ and $(s_1, s_2) \rightarrow \mu$ iff either $s_1 \rightarrow \mu_1$ and $\mu = \mu_1 \times \mathcal{D}_{s_2}$, or $s_2 \rightarrow \mu_2$ and $\mu = \mathcal{D}_{s_1} \times \mu_2$.

3.1 PCTL* and its sublogics

We introduce the syntax of PCTL [11] and PCTL* [4] which are probabilistic extensions of CTL and CTL* respectively. PCTL* over the set AP of atomic propositions are formed according to the following grammar:

$$\varphi ::= a \mid \varphi_1 \land \varphi_2 \mid \neg\varphi \mid \mathbb{P}_{\bowtie q}(\psi)$$
$$\psi ::= \varphi \mid \psi_1 \land \psi_2 \mid \neg\psi \mid \mathsf{X}\,\psi \mid \psi_1 \,\mathsf{U}\, \psi_2$$

where $a \in AP$, $\bowtie \in \{<, >, \leq, \geq\}$, $q \in [0, 1]$. We refer to φ and ψ as (PCTL*) state and path formulae, respectively.

The satisfaction relation $s \models \varphi$ for state formulae is defined in a standard manner for boolean formulae. For probabilistic operator, it is defined by $s \models \mathbb{P}_{\bowtie q}(\psi)$ iff $\forall \sigma. Prob_{\sigma,s}(\{\omega \in Path(s) \mid \omega \models \psi\}) \bowtie q$. The satisfaction relation $\omega \models \psi$ for path formulae is defined exactly the same as for LTL formulae, for example $\omega \models \mathsf{X}\,\psi$ iff $\omega|_1 \models \psi$, and $\omega \models \psi_1 \,\mathsf{U}\, \psi_2$ iff there exists $j \geq 0$ such that $\omega|_j \models \psi_2$ and $\omega|_k \models \psi_1$ for all $0 \leq k < j$.

[4] By standard measure theory this algebra is a *σ-algebra* and all its elements are the measurable sets of paths.

Sublogics. The depth of path formula ψ of PCTL* free of U operator, denoted by $Depth(\psi)$, is defined by the maximum number of embedded X operators appearing in ψ, that is, $Depth(\varphi) = 0$, $Depth(\psi_1 \wedge \psi_2) = \max\{Depth(\psi_1), Depth(\psi_2)\}$, $Depth(\neg\psi) = Depth(\psi)$ and $Depth(\mathsf{X}\,\psi) = 1 + Depth(\psi)$. Then, we let PCTL^{*-} be the sublogic of PCTL* without the until ($\psi_1 \,\mathsf{U}\, \psi_2$) operator. Moreover, PCTL$^{*-}_i$ is a sublogic of PCTL^{*-} where for each ψ we have $Depth(\psi) \leq i$.

The sublogic PCTL is obtained by restricting the path formulae to:

$$\psi ::= \mathsf{X}\,\varphi \mid \varphi_1 \,\mathsf{U}\, \varphi_2 \mid \varphi_1 \,\mathsf{U}^{\leq n}\, \varphi_2$$

Note the bounded until formula does not appear in PCTL* as it can be encoded by nested next operator. PCTL$^-$ is defined in a similar way as for PCTL^{*-}. Moreover we let PCTL^-_i be the sublogic of PCTL$^-$ where only bounded until operator $\varphi_1 \,\mathsf{U}^{\leq j}\, \varphi_2$ with $j \leq i$ is allowed.

Logical equivalence. For a logic $\mathcal{L}$, we say that s and r are $\mathcal{L}$-equivalent, denoted by $s \sim_{\mathcal{L}} r$, if they satisfy the same set of formulae of $\mathcal{L}$, that is $s \models \varphi$ iff $r \models \varphi$ for all formulae φ in $\mathcal{L}$. The logic $\mathcal{L}$ can be PCTL* or one of its sublogics.

3.2 Strong Probabilistic Bisimulation

In this section we introduce the definition of strong probabilistic bisimulation [20]. Let $\{s \to \mu_i\}_{i\in I}$ be a collection of transitions of $\mathcal{P}$, and let $\{p_i\}_{i\in I}$ be a collection of probabilities with $\sum_{i\in I} p_i = 1$. Then $(s, \sum_{i\in I} p_i\mu_i)$ is called a *combined transition* and is denoted by $s \to_{\mathrm{P}} \mu$ where $\mu = \sum_{i\in I} p_i\mu_i$.

Definition 4. *An equivalence relation $\mathcal{R} \subseteq S \times S$ is a strong probabilistic bisimulation iff $s \mathrel{\mathcal{R}} r$ implies that $L(s) = L(r)$ and for each $s \to \mu$, there exists a combined transition $r \to_P \mu'$ such that $\mu \mathrel{\mathcal{R}} \mu'$. We write $s \sim_P r$ whenever there is a strong probabilistic bisimulation $\mathcal{R}$ such that $s \mathrel{\mathcal{R}} r$.*

It was shown in [20] that $\sim_{\mathrm{P}}$ is preserved by $||$, that is, $s \sim_{\mathrm{P}} r$ implies $s \,||\, t \sim_{\mathrm{P}} r \,||\, t$ for any t. Also strong probabilistic bisimulation is sound for PCTL which means that if $s \sim_{\mathrm{P}} r$ then for any state formula φ of PCTL, $s \models \varphi$ iff $r \models \varphi$. But the other way around is not true, i.e. strong probabilistic bisimulation is not complete for PCTL, as illustrated by the following example.

Example 1. Consider again the two PAs in Fig. 1 and assume that $L(s) = L(r)$ and $L(s_1) \neq L(s_2) \neq L(s_3)$. In addition, s_1, s_2, and s_3 only have one transition to themselves with probability 1. The only difference between the left and right automata is that the right automaton has an extra step. It is not hard to see that $s \sim_{PCTL^*} r$. By Definition 4 $s \not\sim_{\mathrm{P}} r$ since the middle transition of r cannot be simulated by s even with combined transition. So we conclude that strong probabilistic bisimulation is not complete for PCTL* as well as for PCTL.

It should be noted that PCTL* distinguishes more states in a PA than PCTL. Refer to the following example.

Example 2. Suppose s and r are given by Fig. 1 where each of s_1, s_2, and s_3 is extended with a transition such that $s_1 \rightarrow \mu_1$ with $\mu_1(s_1) = 0.6$ and $\mu_1(s_4) = 0.4$, $s_2 \rightarrow \mu_2$ with $\mu_2(s_4) = 1$, and $s_3 \rightarrow \mu_3$ with $\mu_3(s_3) = 0.5$ and $\mu_3(s_4) = 0.5$. Here we assume that every state satisfies different atomic propositions except that $L(s) = L(r)$. Then it is not hard to see $s \sim_{PCTL} r$ while $s \not\sim_{PCTL^*} r$. Consider the PCTL* formula $\varphi = \mathbb{P}_{\leq 0.38}(\mathsf{X}(L(s_1) \vee L(s_3)) \wedge \mathsf{X}\mathsf{X}(L(s_1) \vee L(s_3)))$: it holds $s \models \varphi$ but $r \not\models \varphi$. Note that φ is not a well-formed PCTL formula. Indeed, states s and r are PCTL-equivalent.

We have the following theorem:

Theorem 1. *1.* $\sim_{PCTL}$, $\sim_{PCTL^*}$, $\sim_{PCTL^-}$, $\sim_{PCTL_i^-}$, $\sim_{PCTL^{*-}}$, $\sim_{PCTL_i^{*-}}$, *and* $\sim_P$ *are equivalence relations for any* $i \geq 1$.
2. $\sim_P \subseteq \sim_{PCTL^*} \subseteq \sim_{PCTL}$.
3. $\sim_{PCTL^{*-}} \subseteq \sim_{PCTL^-}$.
4. $\sim_{PCTL_1^{*-}} = \sim_{PCTL_1^-}$.
5. $\sim_{PCTL_i^{*-}} \subseteq \sim_{PCTL_i^-}$ *for any* $i > 1$.
6. $\sim_{PCTL} \subseteq \sim_{PCTL^-} \subseteq \sim_{PCTL_{i+1}^-} \subseteq \sim_{PCTL_i^-}$ *for all* $i \geq 0$.
7. $\sim_{PCTL^*} \subseteq \sim_{PCTL^{*-}} \subseteq \sim_{PCTL_{i+1}^{*-}} \subseteq \sim_{PCTL_i^{*-}}$ *for all* $i \geq 0$.

4 A Novel Strong Bisimulation

This section presents our main contribution of the paper: we introduce a novel notion of strong bisimulation and strong branching bisimulation. We shall show that they agree with PCTL and PCTL* equivalences, respectively. As the preparation step we introduce the strong 1-depth bisimulation.

4.1 Strong 1-depth Bisimulation

Definition 5. *A pre-order $\mathcal{R} \subseteq S \times S$ is a strong 1-depth bisimulation if $s \mathcal{R} r$ implies that $L(s) = L(r)$ and for any $\mathcal{R}$-upward-closed set C*

1. *if $s \rightarrow \mu$ with $\mu(C) > 0$, there exists $r \rightarrow \mu'$ such that $\mu'(C) \geq \mu(C)$,*
2. *if $r \rightarrow \mu$ with $\mu(C) > 0$, there exists $s \rightarrow \mu'$ such that $\mu'(C) \geq \mu(C)$.*

We write $s \sim_1 r$ whenever there is a strong 1-depth bisimulation $\mathcal{R}$ such that $s \mathcal{R} r$.

The – though very simple – definition requires only one step matching of the distributions out of s and r. The essential difference to the standard definition is: the quantification of the upward-closed set comes before the transition $s \rightarrow \mu$. This is indeed the key of the new definition of bisimulations. The following theorem shows that $\sim_1$ agrees with $\sim_{PCTL_1^-}$ and $\sim_{PCTL_1^{*-}}$ which is also an equivalence relation:

Lemma 1. $\sim_{PCTL_1^-} = \sim_1 = \sim_{PCTL_1^{*-}}$.

Note that in Definition 5 we consider all the $\mathcal{R}$-upward-closed sets since it is not enough to only consider the $\mathcal{R}$-upward-closed sets in $\{\overline{s_{\mathcal{R}}} \mid s \in S\}$, refer to the following counterexample.

Counterexample 1 *Suppose that there are four absorbing states s_1, s_2, s_3, and s_4 which are assigned with different atomic propositions. Suppose we have two processes s and r such that $L(s) = L(r)$, and $s \to \mu_1$, $s \to \mu_2$, $r \to \nu_1$, $r \to \nu_2$ where $\mu_1(s_1) = 0.5$, $\mu_1(s_2) = 0.5$, $\mu_2(s_3) = 0.5$, $\mu_2(s_4) = 0.5$, $\nu_1(s_1) = 0.5$, $\nu_1(s_3) = 0.5$, $\nu_2(s_2) = 0.5$, $\nu_2(s_4) = 0.5$. If we only consider the $\mathcal{R}$-upward-closed sets in $\{\overline{s_{\mathcal{R}}} \mid s \in S\}$ where $S = \{s, r, s_1, s_2, s_3, s_4\}$, then we will conclude that $s \sim_1 r$, but $r \models \varphi$ while $s \not\models \varphi$ where $\varphi = \mathbb{P}_{\geq 0.5}(\mathsf{X}(L(s_1) \vee L(s_2)))$.*

It turns out that $\sim_1$ is preserved by $||$, implying that $\sim_{PCTL_1^-}$ and $\sim_{PCTL_1^{*-}}$ are preserved by $||$ as well.

Theorem 2. *$s \sim_1 r$ implies that $s \,||\, t \sim_1 r \,||\, t$ for any t.*

Remark 1. We note that for Kripke structure (PA with only Dirac distributions) $\sim_1$ agrees with the usual strong bisimulation by Milner [17].

4.2 Strong Branching Bisimulation

Now we extend the relation $\sim_1$ to strong i-step bisimulations. Then, the intersection of all of these relations gives us the new notion of strong branching bisimulation, which we show to be the same as $\sim_{PCTL}$. Recall Theorem 1 states that $\sim_{PCTL}$ is strictly coarser than $\sim_{PCTL^*}$, which we shall consider in the next section.

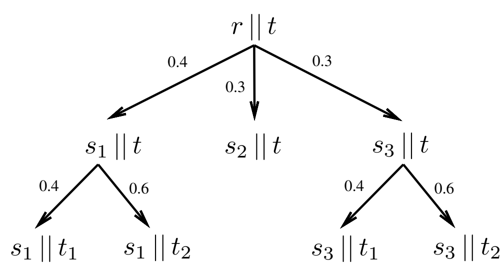

Fig. 2. $\sim_i^b$ is not compositional when $i > 1$

Following the way in [22] we define $Prob_{\sigma,s}(C, C', n, \omega)$ which denotes the probability from s to states in C' via states in C possibly in at most n steps under scheduler σ, where ω is used to keep track of the path and only deterministic schedulers are considered in the following. Formally, $Prob_{\sigma,s}(C, C', n, \omega)$ equals 1 if $s \in C'$, and else if $n > 0 \wedge (s \in C \setminus C')$, then

$$Prob_{\sigma,s}(C, C', n, \omega) = \sum_{r \in supp(\mu')} \mu'(r) \cdot Prob_{\sigma,r}(C, C', n-1, \omega r). \qquad (1)$$

where $\sigma(\omega)(s, \mu') = 1$, otherwise equals 0.

Strong i-depth branching bisimulation is a straightforward extension of strong 1-depth bisimulation, where instead of considering only one immediate step, we consider up to i steps. We let $\sim_1^b = \sim_1$ in the following.

Definition 6. *A pre-order $\mathcal{R} \subseteq S \times S$ is a strong i-depth branching bisimulation if $i > 1$ and $s \,\mathcal{R}\, r$ implies that $s \sim_{i-1}^b r$ and for any $\mathcal{R}$ upward-closed sets C, C',*

1. *if $Prob_{\sigma,s}(C, C', i, s) > 0$ for a scheduler σ, then there exists a scheduler σ' such that $Prob_{\sigma',r}(C, C', i, r) \geq Prob_{\sigma,s}(C, C', i, s)$,*

2. *if* $Prob_{\sigma,r}(C, C', i, r) > 0$ *for a scheduler* σ*, then there exists a scheduler* σ' *such that* $Prob_{\sigma',s}(C, C', i, s) \geq Prob_{\sigma,r}(C, C', i, r)$.

We write $s \sim_i^b r$ *whenever there is a strong* i*-depth branching bisimulation* $\mathcal{R}$ *such that* $s \,\mathcal{R}\, r$*. The strong branching bisimulation* $\sim^b$ *is defined as* $\sim^b = \cap_{i\geq 1} \sim_i^b$.

The following lemma shows that $\sim_i^b$ is an equivalence relation, and moreover, $\sim_i^b$ decreases until a fixed point is reached.

Lemma 2. *1.* $\sim^b$ *and* $\sim_i^b$ *are equivalence relations for any* $i > 1$.
2. $\sim_j^b \subseteq \sim_i^b$ *provided that* $1 \leq i \leq j$.
3. *There exists* $i \geq 1$ *such that* $\sim_j^b = \sim_k^b$ *for any* $j, k \geq i$.

It is not hard to show that $\sim_i^b$ characterizes PCTL_i^-. Moreover, we show that $\sim^b$ agrees with PCTL equivalence.

Theorem 3. $\sim_{PCTL_i^-} = \sim_i^b$ *for any* $i \geq 1$*, and moreover* $\sim_{PCTL} = \sim^b$.

Intuitively, since $\sim_{PCTL_i^-} = \sim_i^b$ decreases with i, for any PA $\sim_i^b$ will eventually converge to PCTL equivalence.

Recall $\sim_1^b$ is compositional by Theorem 2, which unfortunately is not the case for $\sim_i^b$ with $i > 1$. This is illustrated by the following example:

Counterexample 2 $s \sim_i^b r$ *does not imply* $s \,||\, t \sim_i^b r \,||\, t$ *for any* t *generally if* $i > 1$.

We have shown in Example 1 that $s \sim_{PCTL} r$*. If we compose* s *and* r *with* t *where* t *only has a transition to* μ *such that* $\mu(t_1) = 0.4$ *and* $\mu(t_2) = 0.6$*, then it turns out that* $s \,||\, t \not\sim_{PCTL} r \,||\, t$*. Since there exists*

$$\varphi = \mathbb{P}_{\leq 0.34}(true\, \mathsf{U}^{\leq 2}(L(s_1 \,||\, t_2) \vee L(s_3 \,||\, t_1)))$$

such that $s \,||\, t \models \varphi$ *but* $r \,||\, t \not\models \varphi$*, as there exists a scheduler* σ *such that the probability of paths satisfying* ψ *in* $Prob_{\sigma,r}$ *equals 0.36 where* $\psi = (true\, \mathsf{U}^{\leq 2}(L(s_1 \,||\, t_2) \vee L(s_3 \,||\, t_1)))$*. Fig. 2 shows the execution of* r *guided by the scheduler* σ*, and we assume all the states in Fig. 2 have different atomic propositions except that* $L(s \,||\, t) = L(r \,||\, t)$*. It is similar for* $\sim_{PCTL^*}$.

Note that φ *is also a well-formed state formula of* $PCTL_2^-$*, so* $\sim_{PCTL_i^-}$ *as well as* $\sim_i^b$ *are not compositional if* $i \geq 2$.

4.3 Strong Bisimulation

In this section we introduce a new notion of strong bisimulation and show that it characterizes $\sim_{PCTL^*}$. Given a pre-order $\mathcal{R}$, a $\mathcal{R}$-upward-closed cone C_Ω and a measure $Prob$, the value of $Prob(C_\Omega)$ can be computed by summing up the values of all $Prob(C_\omega)$ with $\omega \in \Omega$. We let $\tilde{\Omega} \subseteq \overline{\mathcal{R}}^*$ be a set of $\mathcal{R}$-upward-closed paths, then $C_{\tilde{\Omega}}$ is the corresponding set of $\mathcal{R}$-upward-closed cones, that is, $C_{\tilde{\Omega}} = \cup_{\Omega \in \tilde{\Omega}} C_\Omega$. Define $l(\tilde{\Omega}) = Max\{l(\Omega) \mid \Omega \in \tilde{\Omega}\}$ as the maximum length of Ω in $\tilde{\Omega}$. To compute $Prob(C_{\tilde{\Omega}})$, we cannot sum up the value of each $Prob(C_\Omega)$ such that $\Omega \in \tilde{\Omega}$ as before since we may have a path ω such that $\omega \in \Omega_1$ and $\omega \in \Omega_2$ where $\Omega_1, \Omega_2 \in \tilde{\Omega}$, so we have to remove these duplicate paths and make sure each path is considered once and only once as follows where we abuse the notation and write $\omega \in \tilde{\Omega}$ iff $\exists \Omega.(\Omega \in \tilde{\Omega} \wedge \omega \in \Omega)$:

$$Prob(C_{\tilde{\Omega}}) = \sum_{\omega \in \tilde{\Omega} \wedge \nexists \omega' \in \tilde{\Omega}.\omega' \leq \omega} Prob(C_\omega) \tag{2}$$

Note Equation 2 can be extended to compute the probability of any set of cones in a given measure.

The definition of strong i-depth bisimulation is as follows:

Definition 7. *A pre-order $\mathcal{R} \subseteq S \times S$ is a strong i-depth bisimulation if $i > 1$ and $s \mathcal{R} r$ implies that $s \sim_{i-1} r$ and for any $\tilde{\Omega} \subseteq \overline{\mathcal{R}}^*$ with $l(\tilde{\Omega}) = i$*

1. *if $Prob_{\sigma,s}(C_{\tilde{\Omega}}) > 0$ for a scheduler σ, there exists σ' such that $Prob_{\sigma',r}(C_{\tilde{\Omega}}) \geq Prob_{\sigma,s}(C_{\tilde{\Omega}})$,*
2. *if $Prob_{\sigma,r}(C_{\tilde{\Omega}}) > 0$ for a scheduler σ, there exists σ' such that $Prob_{\sigma',s}(C_{\tilde{\Omega}}) \geq Prob_{\sigma,r}(C_{\tilde{\Omega}})$.*

We write $s \sim_i r$ whenever there is a i-depth strong bisimulation $\mathcal{R}$ such that $s \mathcal{R} r$. The strong bisimulation $\sim$ is defined as $\sim = \cap_{i \geq 1} \sim_i$.

Similar to $\sim_i^b$, the relation $\sim_i$ forms a chain of equivalence relations where the strictness of $\sim_i$ increases as i increases, and $\sim_i$ will converge finally in a PA.

Lemma 3. *1. $\sim_i$ is an equivalence relation for any $i > 1$.*
2. $\sim_j \subseteq \sim_i$ provided that $1 \leq i \leq j$.
3. There exists $i \geq 1$ such that $\sim_j = \sim_k$ for any $j, k \geq i$.

Below we show that $\sim_i$ characterizes $\sim_{PCTL_i^{*-}}$ for all $i \geq 1$, and $\sim$ agrees with PCTL* equivalence:

Theorem 4. *$\sim_{PCTL_i^{*-}} = \sim_i$ for any $i \geq 1$, and moreover, $\sim_{PCTL^*} = \sim$.*

Recall by Lemma 3, there exists $i > 0$ such that $\sim_{PCTL^*} = \sim_i$. For the same reason as strong i-depth branching bisimulation, $\sim_i$ is not preserved by $||$ when $i > 1$.

Counterexample 3 *$s \sim_i r$ does not imply $s \,||\, t \sim_i r \,||\, t$ for any t generally if $i > 1$. This can be shown by using the same arguments as in Counterexample 2.*

4.4 Taxonomy for Strong Bisimulations

Fig. 3 summaries the relationship among all these bisimulations and logical equivalences. The arrow $\rightarrow$ denotes $\subseteq$ and $\nrightarrow$ denotes $\nsubseteq$. We also abbreviate $\sim_{PCTL}$ as PCTL, and it is similar for other logical equivalences. Congruent relations with respect to $||$ operator are shown in circles, and non-congruent in boxes. Segala has considered another strong bisimulation in [20], which can be defined by replacing the $r \rightarrow_{\mathrm{P}} \mu'$ with $r \rightarrow \mu'$ and thus is strictly stronger than $\sim_{\mathrm{P}}$. It is also worth mentioning that all the bisimulations shown in Fig.3 coincide with the strong bisimulation defined in [3] in the DTMC setting which can be seen as a special case of PA (i.e., deterministic probabilistic automata).

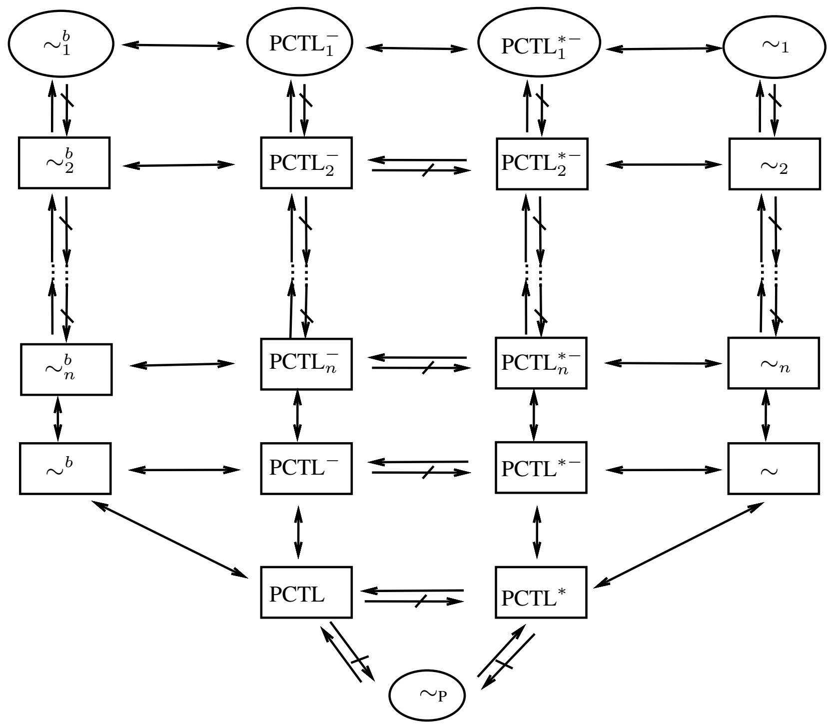

Fig. 3. Relationship of Different Equivalences in Strong Scenario

5 Weak Bisimulations

As in [3] we use $\mathrm{PCTL}_{\setminus \mathsf{X}}$ to denote the subset of PCTL without next operator $\mathsf{X}\,\varphi$ and bounded until $\varphi_1 \,\mathsf{U}^{\leq n}\, \varphi_2$. Similarly, $\mathrm{PCTL}^*_{\setminus \mathsf{X}}$ is used to denote the subset of PCTL* without next operator $\mathsf{X}\,\psi$. In this section we shall introduce weak bisimulations and study the relation to $\sim_{PCTL_{\setminus \mathsf{X}}}$ and $\sim_{PCTL^*_{\setminus \mathsf{X}}}$, respectively. Before this we should point out that $\sim_{PCTL^*_{\setminus \mathsf{X}}}$ implies $\sim_{PCTL_{\setminus \mathsf{X}}}$ but the other direction does not hold. Refer to the following example.

Example 3. Suppose s and r are given by Fig. 1 where each of s_1 and s_3 is attached with one transition respectively, that is, $s_1 \rightarrow \mu_1$ such that $\mu_1(s_4) = 0.4$ and $\mu_1(s_5) = 0.6$, $s_3 \rightarrow \mu_3$ such that $\mu_3(s_4) = 0.4$ and $\mu_3(s_5) = 0.6$. In addition, s_2, s_4 and s_5 only have a transition with probability 1 to themselves, and all these states are assumed to have different atomic propositions. Then $s \ \sim_{PCTL_{\setminus \mathsf{X}}} \ r$ but $s \ \not\sim_{PCTL^*_{\setminus \mathsf{X}}} \ r$, since we have a path formula $\psi = ((L(s) \vee L(s_1)) \,\mathsf{U}\, L(s_5)) \vee ((L(s) \vee L(s_3)) \,\mathsf{U}\, L(s_4))$ such that $s \models \mathbb{P}_{\leq 0.34}\psi$ but $r \not\models \mathbb{P}_{\leq 0.34}\psi$, since there exists a scheduler σ where the probability of path formulae satisfying ψ in $Prob_{\sigma,r}$ is equal to $Prob_{\sigma,r}(ss_1s_5) + Prob_{\sigma,r}(ss_3s_4) = 0.36$. Note ψ is not a well-formed path formula of $\mathrm{PCTL}_{\setminus \mathsf{X}}$.

5.1 Branching Probabilistic Bisimulation by Segala

Before introducing our weak bisimulations, we give the classical definition of branching probabilistic bisimulation proposed in [20]. Given an equivalence relation $\mathcal{R}$, s can evolve into μ by a *branching transition*, written as $s \Rightarrow^{\mathcal{R}} \mu$, iff i) $\mu = \mathcal{D}_s$, or ii) $s \to \mu'$ and

$$\mu = \sum_{r \in (supp(\mu') \cap [s]) \wedge r \Rightarrow^{\mathcal{R}} \mu_r} \mu'(r) \cdot \mu_r + \sum_{r \in supp(\mu') \setminus [s]} \mu'(r) \cdot \mathcal{D}_r$$

where $[s]$ denotes the equivalence class containing s. Stated differently, $s \Rightarrow^{\mathcal{R}} \mu$ means that s can evolve into μ only via states in $[s]$. Accordingly, *branching combined transition* $s \Rightarrow^{\mathcal{R}}_{\mathsf{P}} \mu$ can be defined based on the branching transition, i.e. $s \Rightarrow^{\mathcal{R}}_{\mathsf{P}} \mu$ iff there exists a collection of branching transitions $\{s \Rightarrow^{\mathcal{R}} \mu_i\}_{i \in I}$, and a collection of probabilities $\{p_i\}_{i \in I}$ with $\sum_{i \in I} p_i = 1$ such that $\mu = \sum_{i \in I} p_i \mu_i$.

We give the definition branching probabilistic bisimulation as follows:

Definition 8. *An equivalence relation $\mathcal{R} \subseteq S \times S$ is a branching probabilistic bisimulation iff $s \mathcal{R} r$ implies that $L(s) = L(r)$ and for each $s \to \mu$, there exists $r \Rightarrow^{\mathcal{R}}_{\mathsf{P}} \mu'$ such that $\mu \mathcal{R} \mu'$.*

We write $s \simeq_P r$ whenever there is a branching probabilistic bisimulation $\mathcal{R}$ such that $s \mathcal{R} r$.

The following properties concerning branching probabilistic bisimulation are taken from [20]:

Lemma 4 ([20]).

1. $\simeq_P \subseteq \sim_{PCTL^*_{\setminus \mathsf{X}}} \subseteq \sim_{PCTL_{\setminus \mathsf{X}}}$.
2. $\simeq_P$ *is preserved by* $||$.

5.2 A Novel Weak Branching Bisimulation

Similar to the definition of bounded reachability $Prob_{\sigma,s}(C, C', n, \omega)$, we define the function $Prob_{\sigma,s}(C, C', \omega)$ which denotes the probability from s to states in C' possibly via states in C. Again ω is used to keep track of the path which has been visited. Formally, $Prob_{\sigma,s}(C, C', \omega)$ is equal to 1 if $s \in C'$, $Prob_{\sigma,s}(C, C', \omega)$ is equal to 0 if $s \notin C$, otherwise when $\sigma(\omega)(s, \mu') = 1$,

$$Prob_{\sigma,s}(C, C', \omega) = \sum_{r \in supp(\mu')} \mu'(r) \cdot Prob_{\sigma,r}(C, C', \omega r). \tag{3}$$

The definition of weak branching bisimulation is as follows:

Definition 9. *A pre-order $\mathcal{R} \subseteq S \times S$ is a weak branching bisimulation if $s \mathcal{R} r$ implies that $L(s) = L(r)$ and for any $\mathcal{R}$-upward closed sets C, C'*

1. *if $Prob_{\sigma,s}(C, C', s) > 0$ for a scheduler σ, there exists σ' such that $Prob_{\sigma',r}(C, C', r) \geq Prob_{\sigma,s}(C, C', s)$,*
2. *if $Prob_{\sigma,r}(C, C', r) > 0$ for a scheduler σ, there exists σ' such that $Prob_{\sigma',s}(C, C', s) \geq Prob_{\sigma,r}(C, C', r)$.*

We write $s \approx^b r$ whenever there is a weak branching bisimulation $\mathcal{R}$ such that $s \mathrel{\mathcal{R}} r$.

The following theorem shows that $\approx^b$ is an equivalence relation. Also different from the strong cases where we use a series of equivalence relations to either characterize or approximate $\sim_{PCTL}$ and $\sim_{PCTL^*}$, in the weak scenario we show that $\approx^b$ itself is enough to characterize $\sim_{PCTL_{\setminus \mathsf{X}}}$. Intuitively because in $\sim_{PCTL_{\setminus \mathsf{X}}}$ only unbounded until operator is allowed in path formula which means we abstract from the number of steps to reach certain states.

Theorem 5. *1. $\approx^b$ is an equivalence relation.*
2. $\approx^b \; = \; \sim_{PCTL_{\setminus \mathsf{X}}}$.

As in the strong scenario, $\approx^b$ suffers from the same problem as $\sim_i^b$ and $\sim_i$ with $i > 1$, that is, it is not preserved by $||$.

Counterexample 4 *$s \approx^b r$ does not always imply $s \,||\, t \approx^b r \,||\, t$ for any t. This can be shown in a similar way as Counterexample 2 since the result will still hold even if we replace the bounded until formula with unbounded until formula in Counterexample 2.*

5.3 Weak Bisimulation

In order to define weak bisimulation we consider stuttering paths. Let Ω be a finite $\mathcal{R}$-upward-closed path, then

$$C_{\Omega_{st}} = \begin{cases} C_\Omega & l(\Omega) = 1 \\ \bigcup\limits_{\forall 0 \le i < n. \forall k_i \ge 0} C_{(\Omega[0])^{k_0} \dots (\Omega[n-2])^{k_{n-2}} \Omega[n-1]} & l(\Omega) = n \ge 2 \end{cases} \tag{4}$$

is the set of $\mathcal{R}$-upward-closed paths which contains all stuttering paths, where $\Omega[i]$ denotes the $(i+1)$-th element in Ω such that $0 \le i < l(\Omega)$. Accordingly, $C_{\tilde{\Omega}_{st}} = \bigcup\limits_{\Omega \in \tilde{\Omega}} C_{\Omega_{st}}$ contains all the stuttering paths of each $\Omega \in \tilde{\Omega}$. Given a measure $Prob$, $Prob(\tilde{\Omega}_{st})$ can be computed by Equation (2).

Now we are ready to give the definition of weak bisimulation as follows:

Definition 10. *A pre-order $\mathcal{R} \subseteq S \times S$ is a weak bisimulation if $s \mathrel{\mathcal{R}} r$ implies that $L(s) = L(r)$ and for any $\tilde{\Omega} \subseteq \overline{\mathcal{R}}^*$*

1. *if $Prob_{\sigma,s}(C_{\tilde{\Omega}_{st}}) > 0$ for a scheduler σ, there exists σ' such that $Prob_{\sigma',r}(C_{\tilde{\Omega}_{st}}) \ge Prob_{\sigma,s}(C_{\tilde{\Omega}_{st}})$,*
2. *if $Prob_{\sigma,r}(C_{\tilde{\Omega}_{st}}) > 0$ for a scheduler σ, there exists σ' such that $Prob_{\sigma',s}(C_{\tilde{\Omega}_{st}}) \ge Prob_{\sigma,r}(C_{\tilde{\Omega}_{st}})$.*

We write $s \approx r$ whenever there is a weak bisimulation $\mathcal{R}$ such that $s \mathrel{\mathcal{R}} r$.

The following theorem shows that $\approx$ is an equivalence relation. For the same reason as in Theorem 5, $\approx$ is enough to characterize $\sim_{PCTL^*_{\setminus \mathsf{X}}}$ which gives us the following theorem.

Theorem 6. *1. $\approx$ is an equivalence relation.*

2. $\approx \ = \ \sim_{PCTL^*_{\setminus \mathsf{X}}}$.

Not surprisingly $\approx$ is not preserved by $||$.

Counterexample 5 *$s \approx r$ does not always imply $s \,||\, t \approx r \,||\, t$ for any t. This can be shown by using the same arguments as in Counterexample 4.*

5.4 Taxonomy for Weak Bisimulations

As in the strong cases we summarize the relation of the equivalences in the weak scenario in Fig. 4 where all the denotations have the same meaning as Fig. 3. Compared to Fig. 3, Fig. 4 is much simpler because the step-indexed bisimulations are absent. As in strong cases, here we do not consider the standard definition of branching bisimulation which is a strict subset of $\simeq_{\mathrm{P}}$ and can be defined by replacing $\Rightarrow^{\mathcal{R}}_{\mathrm{P}}$ with $\Rightarrow^{\mathcal{R}}$ in Definition 8. Again not surprisingly all the relations shown in Fig. 4 coincide with the weak bisimulation defined in [3] in DTMC setting.

6 Related Work

For Markov chains, i.e., deterministic probabilistic automata, the logic PCTL characterizes bisimulations, and PCTL without X operator characterizes weak bisimulations [10, 3]. As pointed out in [20], probabilistic bisimulation is sound, but not complete for PCTL for PAs. In the literatures, various extensions of the Hennessy & Milner [12] are considered for characterizing bisimulations. Larsen and Skou [16] considered such an extension of Hennessy-Milner logic, which characterizes bisimulation for *alternating automaton* [16], or labeled Markov processes [8] (PAs but with continuous state space). For probabilistic automata, Jonsson *et al.* [14] considered a two-sorted logic in the Hennessy-Milner style to characterize strong bisimulations. In [13], the results are extended for characterizing also simulations.

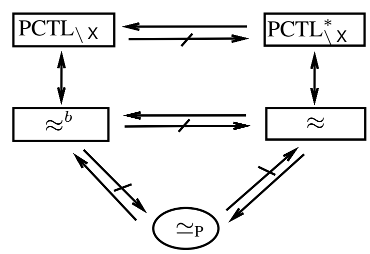

Fig. 4. Relationship of Different Equivalences in Weak Scenario

Weak bisimulation was first defined in the context of PAs by Segala [20], and then formulated for alternating models by Philippou *et al.* [18]. The seemingly very related work is by Desharnais et al. [8], where it is shown that PCTL* is sound and complete with respect to weak bisimulation for alternating automata. The key difference is the model they have considered is not the same as probabilistic automata considered in this paper. Briefly, in alternating automata, states are either nondeterministic like in transition systems, or stochastic like in discrete-time Markov chains. As discussed in [21], a probabilistic automaton can be transformed to an alternating automaton by replacing each transition $s \rightarrow \mu$ by two consecutive transitions $s \rightarrow s'$ and $s' \rightarrow \mu$ where s' is the new inserted state. Surprisingly, for alternating automata, Desharnais et al. have

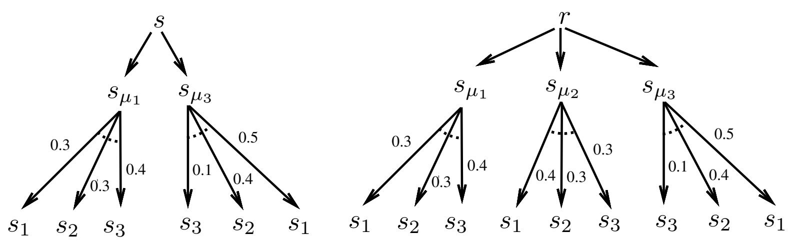

Fig. 5. Alternating Automata

shown that weak bisimulation – defined in the standard manner – characterizes PCTL* formulae. The following example illustrates why it works in that setting, but fails in probabilistic automata.

Example 4. Refer to Fig. 1, we need to add three additional states s_{μ_1}, s_{μ_2}, and s_{μ_3} in order to transform s and r to alternating automata. The resulting automata are shown in Fig. 5. Suppose that s_1, s_2, and s_3 are three absorbing states with different atomic propositions, so they are not (weak) bisimilar with each other, as result s_{μ_1}, s_{μ_2} and s_{μ_3} are not (weak) bisimilar with each other either since they can evolve into s_1, s_2, and s_3 with different probabilities. Therefore s and r are not (weak) bisimilar. Let $\varphi = \mathbb{P}_{\geq 0.4}(\mathsf{X}\, L(s_1)) \wedge \mathbb{P}_{\geq 0.3}(\mathsf{X}\, L(s_2)) \wedge \mathbb{P}_{\geq 0.3}(\mathsf{X}\, L(s_3))$, it is not hard to see that $s_{\mu_2} \models \varphi$ but $s_{\mu_1}, s_{\mu_3} \not\models \varphi$, so $s \models \mathbb{P}_{\leq 0}(\mathsf{X}\, \varphi)$ while $r \not\models \mathbb{P}_{\leq 0}(\mathsf{X}\, \varphi)$. If working with the probabilistic automata, s_{μ_1}, s_{μ_2}, and s_{μ_3} will not be considered as states, so we cannot use the above arguments for alternating automata anymore.

Finally, we want to mention some similarities of $\sim_1$ and notion of metrics studied in [9, 7]. In the definition of $\sim_1$, we choose first the upward-closed set C before the successor distribution to be matched, which is the key for achieving our new notion of bisimulations. This is used in a similar way in defining metrics in [9, 7].

7 Conclusion and Future Work

In this paper we have introduced novel notion of bisimulations for probabilistic automata. They are coarser than the existing bisimulations, and most importantly, we show that they agree with logical equivalences induced by PCTL* and its sublogics. Even in this paper we have not considered actions, it is worth noting that actions can be easily added, and all the results relating (weak) bisimulations hold straightforwardly. On the other side, they are then strictly finer than the logical equivalences, because of the presence of these actions.

As future work, we plan to study decision algorithms for our new (strong and weak) bisimulations, and also extend the work to countable state space.

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A Appendix

A.1 Proofs of Section 3.2

Proof of Theorem 1

Proof. We take $\sim_{PCTL}$ as an example and the others can be proved in a similar way. The reflexivity is trivial. If $s \sim_{PCTL} r$, then we also have $r \sim_{PCTL} s$ since s and r satisfy the same set of formulae, we prove the symmetry of $\sim_{PCTL}$. Now we prove the transitivity, that is, for any s, r, t if we have $s \sim_{PCTL} r$ and $r \sim_{PCTL} t$, then $s \sim_{PCTL} t$. It is also easy, since s and r satisfy the same set of formulae, and r and t satisfy the same set of formulae by $s \sim_{PCTL} r$ and $r \sim_{PCTL} t$, as result $s \models \varphi$ implies $t \models \varphi$ and vice versa for any φ, so $s \sim_{PCTL} t$. We conclude that $\sim_{PCTL}$ is an equivalence relation.

The proof of $\sim_{\mathrm{P}} \subseteq \sim_{PCTL}$ can be found in [20] while the proof of $\sim_{\mathrm{P}} \subseteq \sim_{PCTL^*}$ can be proved in a similar way. $\sim_{PCTL^*} \subseteq \sim_{PCTL}$ is trivial since PCTL is a subset of PCTL*.

The proofs of Clause 3 and 5 are obvious since $\sim_{PCTL^-}$ is a subset of $\sim_{PCTL^{*-}}$ while $\sim_{PCTL_i^-}$ is a subset of $\sim_{PCTL_i^{*-}}$.

We now prove that $\sim_{PCTL_1^{*-}} = \sim_{PCTL_1^-}$. It is sufficient to prove that PCTL$_1^-$ and PCTL$_1^{*-}$ have the same expressiveness. $\sim_{PCTL_1^{*-}} \subseteq \sim_{PCTL_1^-}$ is easy since PCTL$_1^-$ is a subset of PCTL$_1^{*-}$. We now show how formulae of PCTL$_1^{*-}$ can be encoded by formulae of PCTL$_1^-$. It is not hard to see that the syntax of path formulae of PCTL$_1^{*-}$ can be rewritten as:

$$\psi ::= \varphi \mid \mathsf{X}\,\varphi \mid \neg\psi \mid \psi_1 \wedge \psi_2$$

where we replace $\mathsf{X}\,\psi$ with $\mathsf{X}\,\varphi$ since PCTL$_1^{*-}$ only allows path formulae whose depth is less or equal than 1. Since $\neg\,\mathsf{X}\,\varphi = \mathsf{X}\,\neg\varphi$, the syntax can refined further by deleting $\neg\psi$, that is,

$$\psi ::= \varphi \mid \mathsf{X}\,\varphi \mid \psi_1 \wedge \psi_2$$

Then the only left cases we need to consider are $\mathbb{P}_{\bowtie q}(\varphi)$, $\mathbb{P}_{\bowtie q}(\mathsf{X}\,\varphi_1 \wedge \mathsf{X}\,\varphi_2)$, and $\mathbb{P}_{\bowtie q}(\mathsf{X}\,\varphi_1 \wedge \varphi_2)$,

1. $s \models \mathbb{P}_{\geq q}(\varphi)$ iff $s \models \varphi$,
2. $s \models \mathbb{P}_{\geq q}(\mathsf{X}\,\varphi_1 \wedge \mathsf{X}\,\varphi_2)$ iff $s \models \mathbb{P}_{\geq q}(\mathsf{X}(\varphi_1 \wedge \varphi_2))$,
3. $s \models \mathbb{P}_{\geq q}(\mathsf{X}\,\varphi_1 \wedge \varphi_2)$ iff $s \models \varphi_2 \wedge \mathbb{P}_{\geq q}(\mathsf{X}\,\varphi_1)$.

Here we assume that $0 < q \leq 1$, other cases are similar and are omitted.

The proofs of Clauses 6 and 7 are straightforward.

A.2 Proofs of Section 4.1

Proof of Lemma 1

Proof. The proof of the first statement is trivial and is omitted here.

The proof of the second statement is deferred to the proof of Theorem 3 and Theorem 4.

Proof of Theorem 2

Proof. We need to prove that for each $\sim_1$-closed set C, if $s \,||\, t \rightarrow \mu$ such that $\mu(C) > 0$, there exists $r \,||\, t \rightarrow \mu'$ such that $\mu'(C) \geq \mu(C)$ and vice versa. This can be prove by structural induction on $s \,||\, t$ and $r \,||\, t$. By the definition of $||$ operator, if $s \,||\, t \rightarrow \mu$, then either $s \rightarrow \mu_s$ with $\mu = \mu_s \,||\, \mathcal{D}_t$, or $t \rightarrow \mu_t$ with $\mu = \mathcal{D}_s \,||\, \mu_t$. We only consider the case when $\mu = \mu_s \,||\, \mathcal{D}_t$ since the other one is similar. We have known that $s \sim_1 r$, so for each C' if $s \rightarrow \mu_s$ with $\mu_s(C') > 0$, then there exists $r \rightarrow \mu_r$ such that $\mu_r(C') \geq \mu_s(C')$. By induction, if $s' \sim_1 r'$ for $s', r' \in C'$, then $s' \,||\, t \sim_1 r' \,||\, t$. So for each C and $s \,||\, t \rightarrow \mu$ with $\mu(C) > 0$, there exists $r \,||\, t \rightarrow \mu'$ such that $\mu'(C) \geq \mu(C)$.

A.3 Proofs of Section 4.2

In the following, we will use $Sat(\varphi) = \{s \in S \mid s \models \varphi\}$ to denote the set of states which satisfy φ. Similarly, $Sat(\psi) = \{\omega \in Path(s_0) \mid \omega \models \psi\}$ is the set of paths which satisfy ψ.

Proof of Lemma 2

Proof. The proof of the first statement is trivial and is omitted here.

The proof of Clause 2 is straightforward from Definition 7, since $s \sim_j^b r$ implies $s \sim_{j-1}^b r$ when $j > 1$.

It is straightforward from the Definition 6 that $\sim_i^b$ is getting more discriminating as i increases. In a PA only with finite states the maximum number of equivalence classes is equal to the number of states, as result we can guarantee that $\sim_n^b = \sim^b$ where n is the total number of states.

Let $\mathcal{R}$ be an equivalence over S. The set $C \subseteq S$ is said to be $\mathcal{R}$-closed iff $s \in C$ and $s \mathrel{\mathcal{R}} r$ implies $r \in C$. $C_{\mathcal{R}}$ is used to denote the least $\mathcal{R}$-closed set which contains C.

Definition 11. *Two paths $\omega_1 = s_0 s_1 \ldots$ and $\omega_2 = r_0 r_1 \ldots$ are strong i-depth branching bisimilar, written as $\omega_1 \sim_i^b \omega_2$, iff $\omega_1[j] \sim_i^b \omega_2[j]$ for all $0 \leq j \leq i$.*

The $\mathcal{R}$-closed paths can be redefined based on Definition 11. The set Ω of paths is $\sim_i^b$*-closed* if for any $\omega_1 \in \Omega$ and ω_2 such that $\omega_1 \sim_i^b \omega_2$, it holds that $\omega_2 \in \Omega$. Let $\mathcal{B}_{\sim_i^b} = \{\Omega \subseteq \mathcal{B} \mid \Omega \text{ is } \sim_i^b\text{-closed}\}$. By standard measure theory $\mathcal{B}_{\sim_i^b}$ is measurable. The $\sim_i$ for paths can be defined similarly and is omitted here.

Lemma 5. *$s \sim_i^b r$ implies that for each scheduler σ_1 and each $\Omega \in \mathcal{B}_{\sim_i^b}$ such that $Prob_{\sigma,s}(C_\Omega) > 0$ where $\Omega = \bigcup_{0 \leq k < j} C^k C'$ for two $\sim_i^b$-closed sets C, C' with $j \leq i$, there exists σ_2 such that $Prob_{\sigma_2,r}(C_\Omega) \geq Prob_{\sigma_1,s}(C_\Omega)$ and vice versa.*

Proof. Note that by (1) for any $\Omega \in \mathcal{B}_{\sim_i^b}$, if there exists $j < i$ and $\sim_i^b$-closed sets C, C' such that $\Omega = \bigcup_{0 \leq k \leq j} C^k C'$, then $Prob_{\sigma,s}(C, C', j, s) = Prob_{\sigma,s}(C_\Omega)$. The following proof is straightforward from Definition 6.

Proof of Theorem 3

Proof. In order to prove that $s \sim_{PCTL_i^-} r$ implies $s \sim_i^b r$ for any s and r, we need to show that for any $\sim_{PCTL_i^-}$-closed sets C, C', if there exists a scheduler σ such that $Prob_{\sigma,s}(C, C', j, s) > 0$ with $j \leq i$, then there exists a scheduler σ' such that $Prob_{\sigma',r}(C, C', j, r) \geq Prob_{\sigma,s}(C, C', j, s)$ and vice versa provided that $s \sim_{PCTL_i^-} r$. Suppose there are n different equivalence classes in a finite PA. Let φ_{C_i,C_j} be a state formula such that $Sat(\varphi_{C_i,C_j}) \supseteq C_i$ and $Sat(\varphi_{C_i,C_j}) \cap C_j = \emptyset$, here $1 \leq i \neq j \leq n$ and $C_i, C_j \in S/\sim_{PCTL_i^-}$ are two different equivalence classes. Formula like φ_{C_i,C_j} always exists, otherwise there will not exist a formula which is fulfilled by states in C_i, but not fulfilled by states in C_j, that is, states in C_i and C_j satisfy the same set of formulae, this is against the assumption that C_i and C_j are two different equivalence classes. Let $\varphi_{C_i} = \bigwedge_{1 \leq j \neq i \leq n} \varphi_{C_i,C_j}$, it is not hard to see that $Sat(\varphi_{C_i}) = C_i$. For a $\sim_{PCTL_i^-}$-closed set C, it holds

$$\varphi_C = \bigvee_{C' \in S/\sim_{PCTL_i^-} \wedge C' \subseteq C} \varphi_{C'},$$

then $Sat(\varphi_C) = C$. Now suppose $Prob_{\sigma,s}(C, C', j, s) = q > 0$ with $j \leq i$, then we know $s \models \neg \mathbb{P}_{<q}\psi$ where

$$\psi = \varphi_C \,\mathsf{U}^{\leq j}\, \varphi_{C'}.$$

By assumption $r \models \neg \mathbb{P}_{<q}\psi$, so there exists a scheduler σ' such that $Prob_{\sigma',r}(C, C', j, r) \geq q$, that is, $Prob_{\sigma',r}(C, C', j, r) \geq Prob_{\sigma,s}(C, C', j, s)$. The other case is similar and is omitted here.

The proof of $\sim_i^b \subseteq \sim_{PCTL_i^-}$ is by structural induction on the syntax of state formula φ of PCTL$_i^-$ and path formula ψ of PCTL$_i^-$, that is, we need to prove the following two results simultaneously.

1. $s \sim_i^b r$ implies that $s \models \varphi$ iff $r \models \varphi$ for any state formula φ of PCTL$_i^-$.
2. $\omega_1 \sim_i^b \omega_2$ implies that $\omega_1 \models \psi$ iff $\omega_2 \models \psi$ for any path formula ψ of PCTL$_i^-$.

We only consider $\varphi = \mathbb{P}_{\geq q}(\psi)$ here. $s \models \varphi$ iff $\forall \sigma. Prob_{\sigma,s}(\{\omega \mid \omega \models \psi\}) \geq q$. The set Ω of paths satisfying $\psi \in Seq_i^-$, $\Omega = \{\omega \mid \omega \models \psi\}$, is $\sim_i^b$-closed by the induction hypothesis. If $\psi = \mathsf{X}\,\varphi'$, the proof is obvious since $\sim_i^b$ implies $\sim_1^b$. Suppose $\psi = \varphi_1 \,\mathsf{U}^{\leq j}\, \varphi_2$ with $j \leq i$, we need to show that $l(\Omega) \leq i$ and there exists two $\sim_i^b$-closed sets C, C' such that $\Omega = \bigcup_{0 \leq k < j} C^k C'$, this is straightforward by the semantics of $\mathsf{U}^{\leq j}$. By Lemma 5 it follows that for each scheduler σ_1 and each $\Omega \in \mathcal{B}_{\sim_i^b}$ such that $\Omega = \bigcup_{0 \leq k < j} C^k C'$ with $j \leq i$, there exists σ_2 such that $Prob_{\sigma_2,r}(C_\Omega) \geq Prob_{\sigma_1,s}(C_\Omega)$ and vice versa. As result $r \models \varphi$.

To prove $\sim_{PCTL} = \sim^b$ we show first a lemma. We let $\sim^b = \bigcap_{n \geq 1} \sim_n^b$ in the following.

Lemma 6. *$s \sim_{PCTL} r$ iff $s \sim_n^b r$ for any $n \geq 1$, that is, $\sim_{PCTL} = \bigcap_{n \geq 1} \sim_n^b$.*

Proof. The proof is based on the fact that $\varphi_1 \,\mathsf{U}\, \varphi_2 = \varphi_1 \,\mathsf{U}^{\leq \infty}\, \varphi_2$.

The proof of $\sim_{PCTL} = \sim^b$ is straightforward by using Lemma 2 and Lemma 6.

A.4 Proofs of Section 4.3

Proof of Lemma 3

Proof. The proof is similar with the proof of Lemma 2 and is omitted here.

Lemma 7. *$s \sim_i r$ implies that for each scheduler σ_1 and $C_{\tilde{\Omega}}$ such that $\tilde{\Omega} \subseteq \mathcal{B}_{\sim_i}$ such that $l(\tilde{\Omega}) \leq i$, there exists σ_2 such that $Prob_{\sigma_2,r}(C_{\tilde{\Omega}}) \geq Prob_{\sigma_1,s}(C_{\tilde{\Omega}})$ and vice versa.*

Proof. The proof is straightforward from Definition 7.

Proof of Theorem 4

Proof. In order to prove that $s \sim_{PCTL_i^{*-}} r$ implies $s \sim_i r$ for any s and r, we need to show that for any $\tilde{\Omega} \subseteq \overline{\sim_{PCTL_i^{*-}}}^*$ with $l(\tilde{\Omega}) \leq i$, if there exists a scheduler σ such that $Prob_{\sigma,s}(C_{\tilde{\Omega}}) > 0$, then there exists a scheduler σ' such that $Prob_{\sigma',r}(C_{\tilde{\Omega}}) \geq Prob_{\sigma,s}(C_{\tilde{\Omega}})$ and vice versa provided that $s \sim_{PCTL_i^{*-}} r$. Following the way in the proof of Theorem 3, we can construct a formula φ_C such that $Sat(\varphi_C) = C$ where C is a $\sim_{PCTL_i^{*-}}$-closed set. Suppose $\Omega = C_0 C_1 \ldots C_j$ with $j \leq i$, then

$$\psi_\Omega = \varphi_{C_0} \wedge \mathsf{X}(\varphi_{C_1} \wedge \ldots \wedge \mathsf{X}(\varphi_{C_{j-1}} \wedge \mathsf{X}\,\varphi_{C_j}))$$

can be used to characterize Ω, that is, $Sat(\psi_\Omega) = C_\Omega$. Let $\psi = \bigvee_{\Omega \in \tilde{\Omega}} \psi_\Omega$, then $Sat(\psi) = C_{\tilde{\Omega}}$. As result $s \models \neg\mathbb{P}_{<q}\psi$ where $q = Prob_{\sigma,s}(C_{\tilde{\Omega}})$. By assumption $r \models \neg\mathbb{P}_{<q}\psi$, so there exists a scheduler σ' such that $Prob_{\sigma',r}(C_{\tilde{\Omega}}) \geq q$, that is, $Prob_{\sigma',r}(C_{\tilde{\Omega}}) \geq Prob_{\sigma,s}(C_{\tilde{\Omega}})$. The other case is similar and is omitted here.

The proof of $\sim_i \subseteq \sim_{PCTL_i^{*-}}$ is by structural induction on the syntax of state formula φ of PCTL$_i^{*-}$ and path formula ψ of PCTL$_i^{*-}$, that is, we need to prove the following two results simultaneously.

1. $s \sim_i r$ implies that $s \models \varphi$ iff $r \models \varphi$ for any state formula φ of PCTL$_i^{*-}$.
2. $\omega_1 \sim_i \omega_2$ implies that $\omega_1 \models \psi$ iff $\omega_2 \models \psi$ for any path formula ψ of PCTL$_i^{*-}$.

We only consider $\varphi = \mathbb{P}_{\geq q}(\psi)$ here. $s \models \varphi$ iff $\forall\sigma.Prob_{\sigma,s}(\{\omega \mid \omega \models \psi\}) \geq q$. The set $\tilde{\Omega}$ of paths satisfying $\psi \in Seq_i^{*-}$, $\tilde{\Omega} = \{\omega \mid \omega \models \psi\}$, is $\sim_i$-closed by the induction hypothesis, and also $l(\tilde{\Omega}) \leq i$ since the depth of ψ is at most i. By Lemma 7 it follows that for each scheduler σ_1 and each $\tilde{\Omega} \subseteq \mathcal{B}_{\sim_i}$ with $l(\tilde{\Omega}) \leq i$, there exists σ_2 such that $Prob_{\sigma_2,r}(C_{\tilde{\Omega}}) \geq Prob_{\sigma_1,s}(C_{\tilde{\Omega}})$ and vice versa. As result $r \models \varphi$.

To prove the last statement of the theorem, we let $\sim = \bigcap_{n \geq 1} \sim_n$ in the following, and show a lemma first.

Lemma 8. *$s \sim_{PCTL^*} r$ iff $s \sim_n r$ for any $n \geq 1$, that is, $\sim_{PCTL^*} = \bigcap_{n \geq 1} \sim_n$.*

Proof. The proof is similar with the proof of Lemma 6.

The proof is straightforward by using Lemma 3 and Lemma 8.

A.5 Proof of Section 5.1

A.6 Proofs of Section 5.2

Proof of Theorem 5

Proof. 1. The reflexibility of $\approx^b$ is trivial. The symmetry of $\approx^b$ is straightforward from Definition 9. Suppose that $s \approx^b r$ and $r \approx^b t$, then for any $\approx^b$-closed sets C, C', if $Prob_{\sigma,s}(C, C', s) > 0$ for a scheduler σ, there exists σ' such that $Prob_{\sigma',r}(C, C', r) \geq Prob_{\sigma,s}(C, C', s)$. Since we also have $r \approx^b t$, so there exists σ'' such that $Prob_{\sigma'',t}(C, C', t) \geq Prob_{\sigma',r}(C, C', r) \geq Prob_{\sigma,s}(C, C', s)$. Similarly if $Prob_{\sigma,t}(C, C', t) > 0$ for a scheduler σ, there exists σ' such that $Prob_{\sigma',s}(C, C', s) \geq Prob_{\sigma,t}(C, C', t)$. This proves the transitivity of $\approx^b$.

2. In order to prove that $s \sim_{PCTL_{\setminus \mathsf{X}}} r$ implies $s \approx^b r$ for any s and r, we need to show that for any $\sim_{PCTL_{\setminus \mathsf{X}}}$-closed sets C, C', if there exists a scheduler σ such that $Prob_{\sigma,s}(C, C', s) > 0$, then there exists a scheduler σ' such that $Prob_{\sigma',r}(C, C', r) \geq Prob_{\sigma,s}(C, C', s)$ and vice versa provided that $s \sim_{PCTL_{\setminus \mathsf{X}}} r$. Following the way in the proof of Theorem 3, we can construct a formula φ_C such that $Sat(\varphi_C) = C$ where C is a $\sim_{PCTL_{\setminus \mathsf{X}}}$-closed set. Let $\psi = \varphi_C \,\mathsf{U}\, \varphi_{C'}$, then it is not hard to see that $s \models \neg\mathbb{P}_{<q}\psi$ where $q = Prob_{\sigma,s}(C, C', s)$. By assumption $r \models \neg\mathbb{P}_{<q}\psi$, so there exists a scheduler σ' such that $Prob_{\sigma',r}(C, C', r) \geq q$, that is, $Prob_{\sigma',r}(C, C', r) \geq Prob_{\sigma,s}(C, C', s)$. The other case is similar and is omitted here.
The proof of $\approx^b \subseteq \sim_{PCTL_{\setminus \mathsf{X}}}$ is by structural induction on the syntax of state formula φ of PCTL$_{\setminus \mathsf{X}}$ and path formula ψ of PCTL$_{\setminus \mathsf{X}}$, that is, we need to prove the following two results simultaneously.
(a) $s \approx^b r$ implies that $s \models \varphi$ iff $r \models \varphi$ for any state formula φ of PCTL$_{\setminus \mathsf{X}}$.
(b) $\omega_1 \approx^b \omega_2$ implies that $\omega_1 \models \psi$ iff $\omega_2 \models \psi$ for any path formula ψ of PCTL$_{\setminus \mathsf{X}}$.
We only consider $\varphi = \mathbb{P}_{\geq q}(\psi)$ with $\psi = \varphi_1 \,\mathsf{U}\, \varphi_2$ here. $s \models \varphi$ iff $\forall \sigma. Prob_{\sigma,s}(\{\omega \mid \omega \models \psi\}) \geq q$. $\{\omega \mid \omega \models \psi\}$, $Sat(\varphi_1)$, and $Sat(\varphi_2)$ are $\approx^b$-closed by the induction hypothesis, moreover $Prob_{\sigma,s}(\{\omega \mid \omega \models \psi\}) = Prob_{\sigma,s}(Sat(\varphi_1), Sat(\varphi_2), s)$ by Equation (3) for any σ. So for each σ_1 such that $Prob_{\sigma_1,s}(Sat(\varphi_1), Sat(\varphi_2), s) > 0$, there exists σ_2 such that $Prob_{\sigma_2,r}(Sat(\varphi_1), Sat(\varphi_2), r) \geq Prob_{\sigma_1,s}(Sat(\varphi_1), Sat(\varphi_2), s)$ and vice versa. As result $r \models \varphi$.

A.7 Proofs of Section 5.3

Proof of Theorem 6

Proof. 1. The proof is similar with Clause 1 of Theorem 5 and is omitted here.

2. In order to prove that $s \sim_{PCTL^*_{\setminus \mathsf{X}}} r$ implies $s \approx r$ for any s and r, we need to show that for any $\tilde{\Omega} \subseteq \overline{\sim_{PCTL^*_{\setminus \mathsf{X}}}}^*$, if there exists a scheduler σ such that $Prob_{\sigma,s}(C_{\tilde{\Omega}_{st}}) > 0$, then there exists a scheduler σ' such that $Prob_{\sigma',r}(C_{\tilde{\Omega}_{st}}) \geq Prob_{\sigma,s}(C_{\tilde{\Omega}_{st}})$ and vice versa provided that $s \sim_{PCTL^*_{\setminus \mathsf{X}}} r$. Following the way in the proof of Theorem 3, we can construct a formula φ_C such that $Sat(\varphi_C) = C$ where C is a $\sim_{PCTL^*_{\setminus \mathsf{X}}}$-closed set. Let $\psi_\Omega = \varphi_{C_0} \,\mathsf{U} \ldots \varphi_{C_n}$ where $\Omega = C_{C_0 \ldots C_n}$,

then $\psi_{\tilde{\Omega}} = \bigvee_{\Omega\in\tilde{\Omega}} \psi_{\Omega}$. So $s \models \neg\mathbb{P}_{<q}\psi$ where $q = Prob_{\sigma,s}(C_{\tilde{\Omega}_{st}})$ and $\psi = \psi_{\tilde{\Omega}}$. By assumption $r \models \neg\mathbb{P}_{<q}\psi$, so there exists a scheduler σ' such that $Prob_{\sigma',r}(C_{\tilde{\Omega}_{st}}) \geq q$, that is, $Prob_{\sigma',r}(C_{\tilde{\Omega}_{st}}) \geq Prob_{\sigma,s}(C_{\tilde{\Omega}_{st}})$. The other case is similar and is omitted here.

The proof of $\approx \;\subseteq\; \sim_{PCTL^*_{\setminus \mathsf{X}}}$ is by structural induction on the syntax of state formula φ of PCTL$^*_{\setminus \mathsf{X}}$ and path formula ψ of PCTL$^*_{\setminus \mathsf{X}}$, that is, we need to prove the following two results simultaneously.

(a) $s \approx r$ implies that $s \models \varphi$ iff $r \models \varphi$ for any state formula φ of PCTL$^*_{\setminus \mathsf{X}}$.

(b) $\omega_1 \approx \omega_2$ implies that $\omega_1 \models \psi$ iff $\omega_2 \models \psi$ for any path formula ψ of PCTL$^*_{\setminus \mathsf{X}}$.

To make the proof clearer, we rewrite the syntax of PCTL$^*_{\setminus \mathsf{X}}$ as follows which is equivalent to the original definition.

$$\psi ::= \varphi \mid \psi_1 \vee \psi_2 \mid \neg\psi \mid \psi_1 \,\mathsf{U}\, \psi_2$$

We only consider $\varphi = \mathbb{P}_{\geq q}(\psi)$ here. We need to prove that for each σ for each ψ, there exists $\tilde{\Omega} \subseteq \approx^{\infty}$ such that $Prob_{\sigma,s}(\tilde{\Omega}) = Prob\sigma, s(Sat(\psi))$. The proof is by structural induction on ψ as follows:

(a) $\psi = \varphi'$. By induction $Sat(\varphi')$ is $\approx$-closed. Let $\tilde{\Omega} = \{Sat(\varphi')\}$, then $Prob_{\sigma,s}(\tilde{\Omega}) = Prob_{\sigma,s}(Sat(\psi))$.

(b) $\psi = \psi_1 \vee \psi_2$. By induction there exists $\tilde{\Omega}'$ and $\tilde{\Omega}''$ such that $Prob_{\sigma,s}(Sat(\psi_1)) = Prob_{\sigma,s}(C_{\tilde{\Omega}'_{st}})$ and $Prob_{\sigma,s}(Sat(\psi_2)) = Prob_{\sigma,s}(C_{\tilde{\Omega}''_{st}})$. It is not hard to see that $\tilde{\Omega} = \tilde{\Omega}' \cup \tilde{\Omega}''$ will be enough.

(c) $\psi = \psi_1 \,\mathsf{U}\, \psi_2$. By induction there exists $\tilde{\Omega}'$ and $\tilde{\Omega}''$ such that $Prob_{\sigma,s}(Sat(\psi_1)) = Prob_{\sigma,s}(C_{\tilde{\Omega}'_{st}})$ and $Prob_{\sigma,s}(Sat(\psi_2)) = Prob_{\sigma,s}(C_{\tilde{\Omega}''_{st}})$. Let $\tilde{\Omega} = \{\Omega'\Omega'' \mid \Omega' \in \tilde{\Omega}' \wedge \Omega'' \in \tilde{\Omega}''\}$, then $Prob_{\sigma,s}(\tilde{\Omega}) = Prob_{\sigma,s}(Sat(\psi))$.

(d) $\psi = \neg\psi'$. $s \models \mathbb{P}_{\geq q}(\psi)$ iff $s \models \mathbb{P}_{<1-q}(\psi')$, so $\neg\psi'$ can be reduced to another formula without $\neg$ operator.

The following proof is routine and is omitted here.