Count on CFI graphs for #P-hardness*

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Abstract

A *homomorphism* between graphs H and G, possibly with vertex-colors, is a function $f : V(H) \to V(G)$ that preserves colors and edges. Many interesting graph parameters are finite linear combinations $p(\,\cdot\,) = \sum_H \alpha_H \hom(H, \cdot\,)$ of homomorphism counts from fixed pattern graphs H; this includes (induced) subgraph counts for fixed patterns. Interpreting graph parameters as linear combinations of homomorphism counts has proven to be useful in understanding their computational complexity, as it is known that such linear combinations are as hard to evaluate as their hardest terms, whose complexity in turn is governed by the treewidth of the pattern graph. More formally, given oracle access to a linear combination of homomorphism counts p as above and a graph S with coefficient $\alpha_S \neq 0$, it is possible to compute $\hom(S, G)$ for any n-vertex input graph G in $2^{|E(S)|}\mathrm{poly}(s, n)$ time, where s is the maximum size of graphs in the defining linear combination of p. This reduction runs in polynomial time when p and S are fixed or small in comparison to G; this is the relevant setting in several results based on this reduction.

In this paper, we show that a similar reduction can be performed in $\mathrm{poly}(n, s)$ time even if S is part of the input, provided that S has constant maximum degree. Our polynomial-time reduction is based on graph products with Cai–Fürer–Immerman graphs, a novel technique that is likely of independent interest in algorithms and complexity. The new reduction yields #P-hardness results for problems that could previously only be studied under parameterized complexity assumptions such as $\mathsf{FPT} \neq \#\mathsf{W}[1]$, which are a priori stronger than classical assumptions. This includes the problems $\#\mathrm{Hom}(\mathcal{H})$, $\#\mathrm{Sub}(\mathcal{H})$ and $\#\mathrm{Ind}(\mathcal{H})$ for fixed graph classes $\mathcal{H}$ satisfying natural polynomial-time enumerability conditions, which ask to count homomorphisms from H to G or (induced) subgraph copies of H in G, given as input a graph $H \in \mathcal{H}$ and a general graph G.

1 Introduction

In this paper, graphs G are undirected and may be vertex-colored, i.e., they may be given with a coloring $c : V(G) \to C$ for some set C that is specified from the context. We write $G \simeq G'$ if two graphs G and G' admit a color-preserving isomorphism. Uncolored graphs can be viewed as colored graphs with a single color. A graph parameter is a function p that maps graphs into $\mathbb{Q}$ such that $p(G) = p(G')$ holds for isomorphic graphs $G \simeq G'$. Examples include the functions that map a graph G to its maximum degree or maximum clique size. We focus on graph parameters that admit obvious interpretations as weighted counts of objects in graphs, e.g., the number of triangles.

1.1 Graph motif parameters Many interesting graph parameters are *homomorphism counts* or finite linear combinations thereof. A homomorphism from a graph H to a graph G is a function $f : V(H) \to V(G)$ such that $uv \in E(H)$ implies $f(u)f(v) \in E(G)$. If H and G are colored, then the colors of v and $f(v)$ must agree for all $v \in V(H)$. For any fixed graph H, let $\hom(H, \cdot\,)$ be the graph parameter that maps graphs G to the number $\hom(H, G)$ of homomorphisms from H to G. Note that H is fixed and only G is part of the input.

The functions $\hom(H, \cdot\,)$ are known to be linearly independent [34, Proposition 5.43]: If the all-zeros graph parameter is expressed as a finite linear combination of these functions for pairwise non-isomorphic graphs, then this linear combination must be trivial. Thus, no two distinct finite linear combinations of homomorphism counts yield the same graph parameter. This enables the following definition:

Definition 1.1. (see [15]) *A* graph motif parameter *p is a graph parameter*

$$p(\,\cdot\,) = \sum_{H \in \mathcal{H}} \alpha_H \hom(H, \cdot\,) \tag{1.1}$$

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for a finite set of pairwise non-isomorphic graphs $\mathcal{H}$ and coefficients $\alpha_H \in \mathbb{Q}$ for $H \in \mathcal{H}$. We call the right-hand side of (1.1) the hom-expansion *of p. Given a graph H, we write $H \lhd p$ if $H \in \mathcal{H}$ and $\alpha_H \neq 0$.*

Besides the functions $\mathrm{hom}(H, \cdot)$ for fixed graphs H, other examples for graph motif parameters include the functions $\mathrm{sub}(H, \cdot)$ that count subgraphs isomorphic to fixed graphs H:

FACT 1.1. For any graph H, the function $\mathrm{sub}(H, \cdot)$ is a graph motif parameter, see [34, (5.18)]. We have $F \lhd \mathrm{sub}(H, \cdot)$ iff F is a *quotient* of H, i.e., if F can be obtained from H by repeated identifications of vertices of the same color. For later use, we remark that the *sign* of the coefficient α_F (but not the coefficient itself) in the hom-expansion of $\mathrm{sub}(H, \cdot)$ is $(-1)^{|V(H)|-|V(F)|}$.

A similar statement holds for the functions $\mathrm{ind}(H, \cdot)$ that count induced subgraphs isomorphic to H.

FACT 1.2. For any graph H, the function $\mathrm{ind}(H, \cdot)$ is a graph motif parameter, see [34, (5.20)]. Moreover, any graph $F \lhd \mathrm{ind}(H, \cdot)$ satisfies $|V(F)| \leq |V(H)|$. The graphs F with $F \lhd \mathrm{ind}(H, \cdot)$ and $|V(F)| = |V(H)|$ are precisely the graphs obtained by adding edges to H.

Many counting problems related to small patterns can be expressed naturally as linear combinations of (induced) subgraph counts, and Facts 1.1 and 1.2 then imply that they are graph motif parameters. Examples include the numbers of vertex-subsets inducing a graph isomorphic to some graph among some fixed finite set of graphs $\mathcal{P}$ [41, 40, 30, 29, 31], or variants of the Tutte polynomial that only sum over subgraphs with a fixed number of edges [42].

1.2 The known complexity monotonicity To prepare the statement of our main result, we collect known facts on the complexity of graph motif parameters in this subsection. Firstly, any fixed graph motif parameter p can be evaluated in polynomial time on n-vertex input graphs G: Using brute-force, we can evaluate $\mathrm{hom}(H,G)$ for all graphs $H \lhd p$ in time $O(n^s)$ with $s = \max_{H \lhd p} |V(H)|$, and then we can compute the required finite linear combination with a finite number of arithmetic operations. Having observed the polynomial-time solvability of p, one may then (i) investigate the precise exponent in this polynomial running time using fine-grained complexity assumptions, or (ii) adopt an approach rooted in "coarse-grained" parameterized complexity: Given an infinite family of graph motif parameters $\{p_i\}_{i \in I}$ for some index set I, can we evaluate $p_i(G)$ in polynomial time on input $i \in I$ and G? This question captures well-studied counting problems like $\#\mathrm{Sub}(\mathcal{H})$ for fixed graph classes $\mathcal{H}$, which asks to count occurrences of an input graph $H \in \mathcal{H}$ in general graphs G.

Previous works developed an approach for answering such questions [15], which has been dubbed the *complexity monotonicity framework* and was applied successfully to various counting problems [39, 41, 42, 40, 23, 37, 15, 4]. The framework leverages the remarkable fact that evaluating any fixed linear combination of homomorphism counts is asymptotically as hard as evaluating its hardest terms.[1] The upper bound in this statement is trivial, while the lower bound follows from a reduction which computes $\mathrm{hom}(S,G)$ for graphs $S \lhd p$ and G with access to an oracle for $p(\cdot)$. The running time of this known reduction is polynomial in $|V(G)|$ and in the maximum size of graphs $H \lhd p$, but exponential in the size of the pattern S whose homomorphism count is to be extracted.

THEOREM 1.1. (VARIANTS SHOWN IN [15, 8]) *There is an algorithm for the following problem: Output the value $\mathrm{hom}(S,G)$ when given*

- *as input a number $s \in \mathbb{N}$, graphs S and G, and*
- *oracle access for a graph motif parameter p with $\max_{H \lhd p} |V(H)| \leq s$ and $S \lhd p$.*

The running time of the algorithm is bounded by $2^{|E(S)|} \cdot \mathrm{poly}(|V(G)|, s)$, and the oracle is only called on graphs with $\leq s \cdot |V(G)|$ vertices.

[1] Let us note that other natural bases of graph motif parameters do not share such favorable properties. For example, counting k-cliques is hard [35, 9], but the sum of induced subgraph counts of all k-vertex graphs, which includes k-cliques, can be evaluated trivially.

Theorem 1.1 reduces complexity-theoretic questions about graph motif parameters p to questions about homomorphism counts $\hom(H, \cdot)$ for graphs $H \triangleleft p$. These problems are well-understood in various computational models [32] and under complexity-theoretic assumptions. To set the stage for our new result, we first discuss some known "fine-grained" results (based on the exponential-time hypothesis) and "coarse-grained" variants (based on parameterized complexity assumptions).

Fine-grained results for individual graph motif parameters The counting exponential-time hypothesis #ETH [28, 21] postulates that counting satisfying assignments to Boolean formulas on n variables requires $\exp(\Omega(n))$ time. Under this hypothesis, it is known that the complexity of counting homomorphisms is essentially determined by the *treewidth* $\mathrm{tw}(H)$, a graph width measure that is polynomially related to the maximum $r \in \mathbb{N}$ such that H contains an $r \times r$ grid minor [18, 7].

THEOREM 1.2. ([19, 35, 15]) *For any graph H of treewidth t, the graph parameter $\hom(H, \cdot)$ can be evaluated in time $O(n^{t+1})$ on n-vertex graphs. Assuming #ETH, there exists a constant $c > 0$ such that, for any graph H of treewidth t, the graph parameter $\hom(H, \cdot)$ cannot be evaluated in time $O(n^{ct/\log t})$.*

Together with Theorem 1.1, we obtain the following corollary.

COROLLARY 1. ([15]) *Let p be a graph motif parameter and let $q = \max_{H \triangleleft p} \mathrm{tw}(H)$. Then p can be evaluated in time $O(n^{q+1})$. Assuming #ETH, there is a constant $c > 0$, independent of p, such that p cannot be evaluated in time $O(n^{cq/\log q})$.*

This allows us to study the complexity of a graph motif parameter p by determining the maximum treewidth of graphs $H \triangleleft p$ in the hom-expansion of p. We show this exemplarily for the case of subgraph counts.

EXAMPLE 1. *Following [15], let us investigate the complexity of $\mathrm{sub}(H, \cdot)$ for fixed H. By Fact 1.1, we have $F \triangleleft \mathrm{sub}(H, \cdot)$ iff F is a quotient of H, so it suffices to determine the maximum treewidth q that can be attained by quotients of H. It can be verified that $q = \Theta(k)$, so it follows by Corollary 1 that $\mathrm{sub}(H, \cdot)$ can be evaluated in time $n^{O(k)}$ and, assuming #ETH, not in time $n^{o(k/\log k)}$.*

A similar argument shows that $\mathrm{ind}(H, \cdot)$ with $|V(H)| = k$ cannot be solved in time $n^{o(k)}$ under #ETH: We use that the complete graph K_k satisfies $K_k \triangleleft \mathrm{ind}(H, \cdot)$ by Fact 1.2 and that $\hom(K_k, \cdot)$ cannot be solved in time $n^{o(k)}$ under #ETH, see [9]. Similar results are known for other graph motif parameters that count restricted types of homomorphisms [39], vertex-induced subgraphs inducing monotone properties [41, 40, 23], edge-induced subgraphs satisfying minor-closed properties [37, 42], and for a parameterized variant of the Tutte polynomial [42].

Coarse-grained results for families of graph motif parameters In this paper, rather than focusing on *individual* graph motif parameters, we study *families* of graph motif parameters [19, 11, 16]. In previous works, such problems were addressed in the framework of *parameterized complexity theory* [18, 26, 24], where inputs to computational problems come in the form (x, k) with a traditional input x and a *parameter* k that measures some notion of complexity in x. Such a problem is *fixed-parameter tractable* if it can be solved in time $f(k)n^{O(1)}$ for some computable function f.

Graph motif parameters naturally lead to a parameterized problem [15]: Given as input coefficients $\alpha_1, \dots, \alpha_t \in \mathbb{Q}$, graphs $H_1, \dots, H_t$, and an additional graph G, evaluate $p(G) = \sum_{i=1}^{t} \alpha_i \hom(H_i, G)$. The parameter is $\max_i |V(H_i)|$. This problem is hard for the parameterized complexity class #W[1], which is the parameterized analogue of NP, and more generally, a parameterized problem is #W[1]-hard if counting k-cliques can be reduced to it via so-called parameterized reductions.

To obtain a more refined view on the above evaluation problem for graph motif parameters, we can fix a recursively enumerable family $\mathcal{A}$ of graph motif parameters (represented as lists of coefficients and graphs) and consider the problem under the promise that $p \in \mathcal{A}$. The resulting set of problems includes the problems $\#\mathrm{Hom}(\mathcal{H})$ for recursively enumerable graph classes $\mathcal{H}$: On input $H \in \mathcal{H}$ and G, determine $\hom(H, G)$, parameterized by $|V(H)|$. This problem is known to be polynomial-time solvable if the maximum treewidth among graphs in $\mathcal{H}$ is bounded by a constant, while it is #W[1]-hard otherwise [19]. Together with the complexity monotonicity for linear combinations of homomorphism counts (Theorem 1.1), a classification for the graph motif parameter evaluation problem follows:

THEOREM 1.3. ([15]) *Let $\mathcal{A}$ be a recursively enumerable family of graph motif parameters. If the maximum treewidth of graphs $F \lhd p$ for $p \in \mathcal{A}$ is unbounded, then evaluating $p(G)$ on input $p \in \mathcal{A}$ and G is #W[1]-hard. Otherwise, the problem is fixed-parameter tractable.*

For example, consider the problem $\#\mathrm{Sub}(\mathcal{H})$ for fixed recursively enumerable graph classes $\mathcal{H}$, defined analogously to $\#\mathrm{Hom}(\mathcal{H})$, but asking to count subgraphs isomorphic to an input graph $H \in \mathcal{H}$ rather than homomorphisms from H. By combining Theorem 1.3 and Fact 1.1, this problem is #W[1]-hard if the vertex-cover number of graphs in $\mathcal{H}$ is unbounded. Otherwise, the problem is fixed-parameter tractable, and actually even polynomial-time solvable [46, 16]. Likewise, the problem $\#\mathrm{Ind}(\mathcal{H})$ is #W[1]-hard if $\mathcal{H}$ is infinite and polynomial-time solvable otherwise. While classifications for these parameterized problems were known before the introduction of the complexity monotonicity framework [19, 11, 16], that framework enabled significantly simpler proofs.

1.3 Our result: Complexity monotonicity via polynomial-time reductions The main technical result of the present paper is a *polynomial-time variant* of Theorem 1.1, the central reduction of the complexity monotonicity framework. As in Theorem 1.1, we obtain a Turing reduction that computes $\hom(S, \cdot)$ for a graph S with an oracle for a graph motif parameter p with $S \lhd p$, but our new reduction runs in polynomial time when S has bounded degree. Additionally, the new reduction allows us to reduce homomorphism counts of *colorful* (i.e., bijectively colored) graphs to the evaluation of an *uncolored* graph parameter. This may appear like a technicality, but it is actually crucial, because #P-hardness of counting colorful homomorphisms from S can be shown relatively straightforwardly, while the required #P-hardness proofs for uncolored graphs S were not known prior to this paper.

In the following, we call a colored graph H *colorful* if its coloring is a bijection. Unless stated otherwise, we assume that the color set of a colorful graph H is $V(H)$. We also write H° for the uncolored graph underlying H, and we note that $\hom(H, G) \neq \hom(H^\circ, G^\circ)$ in general.

THEOREM 1.4. *There is an algorithm for the following problem: Output the number* $\hom(S, G)$ *of (color-preserving) homomorphisms from S to G when given*

- *as input a number $s \in \mathbb{N}$, colorful graphs S and T with $S \subseteq T$, a colored graph G, and*
- *oracle access for a graph motif parameter p with $\max_{H \lhd p} |V(H)| \leq s$ such that $T^\circ \lhd p$ and either*
  - *(a) $S = T$, or*
  - *(b) all graphs on $|V(T)|$ vertices in the hom-expansion of p have the same sign, or*
  - *(c) no proper edge-subgraph of T° is contained in the hom-expansion of p.*

The running time of the algorithm is bounded by $4^{\Delta(S)} \cdot \mathrm{poly}(|V(G)|, s)$.

There are several differences between the known reduction and our new variant; let us elaborate on these. Most importantly, the exponential dependence in the running time of Theorem 1.4 is confined to the maximum degree $\Delta(S)$ rather than $|E(S)|$. In particular, the reduction runs in polynomial time if S has constant maximum degree, even when the sizes of S and G are comparable.

Note also that we reduce from S but only require a supergraph $T \supseteq S$ with $T^\circ \lhd p$, subject to one of the three technical conditions of the theorem. This is a useful strengthening, since any graph T of large treewidth contains a large *wall*, which is a subgraph $S \subseteq T$ of large treewidth and maximum degree 3. By invoking the theorem with S instantiated to this wall, we can avoid an exponential running time even for large-degree graphs T. The three technical conditions in the theorem capture different types of "padding" that are performed in the reduction to discard vertices and edges from T not present in S.

Moreover, we reduce $\hom(S, \cdot)$ for colorful graphs S to evaluating an uncolored graph parameter p. As mentioned before, this is already interesting when $p = \hom(S^\circ, \cdot)$ is the uncolored homomorphism count from S°, as the previously known techniques achieving this reduction require $2^{|E(S)|} n^{O(1)}$ time. Because of this running time, comprehensive classifications for $\#\mathrm{Hom}(\mathcal{H})$ in terms of polynomial-time and #P-hard classes $\mathcal{H}$ were not attainable with previous techniques.[2]

[2]For comparison, it is somewhat easier to reduce $\hom(S, \cdot)$ for a colorful graph S to $\hom(R, \cdot)$ for an uncolored graph R derived

Implications for families of graph motif parameters Our new Theorem 1.4 implies that several important families of graph motif parameters are #P-hard. This shows that techniques developed for the complexity monotonicity framework in the context of parameterized complexity can also yield polynomial-time reductions and #P-hardness results instead of #W[1]-hardness results.

A first application concerns the problems #Hom($\mathcal{H}$) mentioned above. The original complexity classification for these problems is a classical result in parameterized complexity and database theory, and it showed #W[1]-hardness for recursively enumerable graph classes $\mathcal{H}$ of unbounded treewidth [19]. Together with a simple *polynomial-time* (rather than merely fixed-parameter tractable) algorithm for graph classes of bounded treewidth, this classifies the *polynomial-time* solvable cases of #Hom($\mathcal{H}$) under the assumption FPT $\neq$ #W[1], subject to the condition that $\mathcal{H}$ is recursively enumerable.

Our new theorem imposes a natural polynomial-time enumerability condition on $\mathcal{H}$ and classifies the polynomial-time solvable cases of #Hom($\mathcal{H}$) under the *weaker* assumption FP $\neq$ #P. To state the criterion, let us define an $r \times r$ *wall* to be any graph obtained from the $r \times r$ square grid by deleting even-indexed edges from even-indexed columns and odd-indexed edges from odd-indexed columns, and then subdividing edges arbitrarily often. Any graph of treewidth t contains an $r \times r$ wall subgraph S with $r \in \Omega(t^{1/100})$, which can be found with a randomized polynomial-time algorithm [7].[3]

THEOREM 1.5. *If a graph class $\mathcal{H}$ admits a polynomial-time algorithm that outputs a graph $H \in \mathcal{H}$ of treewidth $\geq t$ on input $t \in \mathbb{N}$ in unary, then #Hom($\mathcal{H}$) is #P-complete under randomized polynomial-time reductions. Moreover, if there even exists a polynomial-time algorithm that outputs H together with a $t \times t$ wall subgraph in H, then #Hom($\mathcal{H}$) is #P-complete under* deterministic *polynomial-time reductions.*

The theorem is shown by combining our new Theorem 1.4 with a standard #P-hardness proof for counting homomorphisms from colorful walls. Note that a polynomial-time constructibility condition is required in view of pathological classes like $\mathcal{P} = \{K_k \cup \overline{K_{2^k}} \mid k \in \mathbb{N}\}$: Any polynomial-time algorithm for #Hom($\mathcal{P}$) would render the k-clique problem fixed-parameter tractable and thus imply FPT = #W[1], which in turn refutes #ETH. At the same time, the problem #Hom($\mathcal{P}$) is unlikely to be #P-hard: Since it can be solved in quasi-polynomial time, its #P-hardness would refute #ETH among other complexity-theoretic assumptions. Note that $\mathcal{P}$ does not satisfy the conditions of the theorem.

A result similar to Theorem 1.5 can also be shown for counting *subgraphs*. The original complexity classification [16, 15] established #W[1]-hardness for recursively enumerable graph classes $\mathcal{H}$ of unbounded vertex-cover number. We can show a similar result with #P-hardness taking the role of #W[1]-hardness: If H has a matching on t edges, then every graph S on t edges (in particular, a large wall) is contained as a subgraph in some quotient T of H, and such a quotient T and subgraph S in T can be found efficiently. By Fact 1.1, we have $T \lhd p$. Then Theorem 1.4 allows us to show the following:

THEOREM 1.6. *If a graph class $\mathcal{H}$ admits a polynomial-time algorithm that outputs a graph $H \in \mathcal{H}$ of vertex-cover number $\geq t$ on input $t \in \mathbb{N}$ in unary, then #Sub($\mathcal{H}$) is #P-complete.*

Like Theorem 1.5 for homomorphism counts, the proof of this theorem also starts from the #P-hardness of counting colorful walls, and it invokes Theorem 1.4 to reduce this problem to #Sub($\mathcal{H}$). As opposed to Theorem 1.5 however, the relevant walls in the quotients of H can be found *deterministically* in polynomial time.

Finally, we can also classify induced subgraph counting problems. For these problems, we rely on polynomial-time algorithms for finding large clique minors in graphs of large average degree to find large walls in $H \in \mathcal{H}$ or the complement of H.

THEOREM 1.7. *If a graph class $\mathcal{H}$ admits a polynomial-time algorithm that outputs a graph $H \in \mathcal{H}$ on $\geq t$ vertices on input $t \in \mathbb{N}$ in unary, then #Ind($\mathcal{H}$) is #P-complete.*

from S, say, by adding gadgets to S that allow for distinguishing colors [5]. In the same spirit, there are elementary hardness proofs for deciding the existence of colorful $k+k$ bicliques and $k \times k$ grids in colored graphs, but the known hardness results for the *uncolored* version of these problems require significant technical effort [10, 33].

[3]Randomization does not seem to be inherently required to find the relevant wall subgraph, but no deterministic polynomial-time algorithm is known to the best of our knowledge. Consequently, the theorem below requires randomized rather than deterministic reductions when no wall is given along with H.

Let us remark that the main technical effort in this paper lies in proving Theorem 1.4, while Theorems 1.5–1.7 follow relatively straightforwardly from known results. Also note that the #P-hardness results from Theorems 1.5–1.7 are complemented by straightforward polynomial-time algorithms when the relevant graph parameters (treewidth, vertex-cover number, size) are bounded in a class $\mathcal{H}$.

1.4 Proof outline This section already covers large parts of the proof of Theorem 1.4 and therefore introduces some of the concepts and notation used later, especially in Section 4. The proof is built around several *filters* that allow us to progressively narrow the hom-expansion of a graph motif parameter p down to a desired graph S. In fact, the previously known Theorem 1.1 can also be shown using such filters. We first outline a proof of Theorem 1.1 along these lines and then describe the modification that yields the new Theorem 1.4. In the argument, the following filters will be relevant:

1. A *cardinality filter* allows us to restrict the hom-expansion of p to graphs with a desired number of vertices. Such a filter can be constructed via polynomial interpolation arguments that are standard in counting complexity.

2. Conditioned on the cardinality restriction, a *color-surjectivity filter* restricts the hom-expansion further to graphs that include S as a subgraph. Such filters can be realized using the inclusion-exclusion principle, incurring a running time overhead of $2^{|E(S)|}$. Our main contribution in this paper is a more efficient and possibly surprising reduction based on *Cai-Fürer-Immerman graphs*. These graphs were originally used in the context of graph isomorphism and finite model theory [6].

In the remainder of this section, we first introduce the general notion of filters more formally. Then we elaborate on the two concrete filters described above and sketch previously known constructions thereof. Finally, we describe how to obtain color-surjectivity filters from CFI graphs.

General facts on filters Filters are implemented as *quantum graphs*, which were introduced by Lovasz [34] in the context of graph algebras. They are linear combinations of graphs but can often be treated like graphs, both in conceptual and computational terms. More precisely, a *quantum graph* is a formal linear combination

$$\mathbf{G} = \sum_{i=1}^{t} \alpha_i \, G_i$$

with *coefficients* $\alpha_1, \ldots, \alpha_t \in \mathbb{Q}$ and *constituent graphs* $G_1, \ldots, G_t$, either all uncolored or all colored. We assume the constituent graphs to be pairwise non-isomorphic, as we could otherwise collect isomorphic graphs and represent them by one single graph whose coefficient is the sum of the collected coefficients. Graph parameters p are linearly extended to quantum graphs via

$$p(\mathbf{G}) := \sum_{i=1}^{t} \alpha_i \, p(G_i).$$

Based on these definitions, *filters* are quantum graphs $\mathbf{F}$ such that homomorphism counts into $\mathbf{F}$ recognize the "good" graphs $\mathcal{G}$ among some set of "filterable" graphs $\mathcal{F}$.

Definition 1.2. *A quantum graph* $\mathbf{F}$ filters *a set of graphs* $\mathcal{G}$ *out of a set of graphs* $\mathcal{F} \supseteq \mathcal{G}$ *if, for all* $H \in \mathcal{F}$,

$$\hom(H, \mathbf{F}) = \begin{cases} 1 & H \in \mathcal{G}, \\ 0 & H \in \mathcal{F} \setminus \mathcal{G}. \end{cases}$$

Note that no statement is made about graphs $H \notin \mathcal{F}$.

Example 2. *The quantum graph* $\mathbf{F} = \frac{1}{24} K_4 - \frac{1}{6} K_3$ *filters* $\mathcal{G} = \{K_4\}$ *out of* $\mathcal{F} = \{K_3, K_4\}$, *since*

$$\hom(K_4, \mathbf{F}) = \frac{1}{24} 24 - \frac{1}{6} 0 = 1,$$
$$\hom(K_3, \mathbf{F}) = \frac{1}{24} 24 - \frac{1}{6} 6 = 0.$$

No guarantees are made for graphs $H \notin \mathcal{F}$. For example, we have $K_2 \notin \mathcal{F}$ and

$$\hom(K_2, \mathbf{F}) = \frac{1}{24}12 - \frac{1}{6}6 = -\frac{1}{2}.$$

Given a graph motif parameter p on support $\mathcal{F}$ and a quantum graph $\mathbf{F}$ that filters some set $\mathcal{G}$ out of $\mathcal{F}$, graph products with $\mathbf{F}$ will allow us to restrict the hom-expansion of p to $\mathcal{G}$. These products are defined as follows, see also [34, Chapter 3.3] for a variant involving uncolored graphs.

Definition 1.3. *Given a graph G with coloring $c : V(G) \to C$ and $i \in C$, write $V_i(G)$ for the i-colored vertices in G. For graphs G and X with colors C, the* tensor product *$G \otimes X$ is the graph on colors C with*

- *vertex set $V_i(G \otimes X) = V_i(G) \times V_i(X)$ for every color $i \in C$, and*
- *an edge between (u_G, u_X) and (v_G, v_X) iff $u_G v_G \in E(G)$ and $u_X v_X \in E(X)$.*

For quantum graphs $\mathbf{G} = \sum_i \alpha_i G_i$ and $\mathbf{X} = \sum_j \beta_j X_j$, we set $\mathbf{G} \otimes \mathbf{X} = \sum_{i,j} (\alpha_i \beta_j) (G_i \otimes X_j)$.

A very useful algebraic relation holds for homomorphism counts and tensor products, see [34, (5.30)] for uncolored graphs.

Fact 1.3. *Given a graph F and quantum graphs $\mathbf{G}$ and $\mathbf{X}$, either all colored or all uncolored, we have*

$$\hom(F, \mathbf{G} \otimes \mathbf{X}) = \hom(F, \mathbf{G}) \cdot \hom(F, \mathbf{X}).$$

Proof. We first prove the statement for graphs G and X. Any homomorphism $f : V(F) \to V(G \otimes X)$ induces a unique pair of homomorphisms (f_G, f_X) from F to G and X, where f_G and f_X are obtained by projecting all image vertices $f(v)$ for $v \in V(F)$ to the first or second component, respectively. Likewise, any such pair (f_G, f_X) induces a unique homomorphism from F to $G \otimes X$ by pairing the image vertices $f_G(v)$ and $f_X(v)$ for all $v \in V(F)$. The statement for quantum graphs then follows by distributivity. □

Fact 1.3 implies that we can use tensor products with filters to restrict the hom-expansion of a graph motif parameter from a set $\mathcal{F}$ to a desired set $\mathcal{G}$. The following corollary encapsulates this.

Corollary 2. *Let $p(\cdot) = \sum_{F \in \mathcal{F}} \alpha_F \hom(F, \cdot)$ be a graph motif parameter with a finite set of graphs $\mathcal{F}$. Let $\mathbf{F}_1, \ldots, \mathbf{F}_b$ for $b \in \mathbb{N}$ be quantum graphs with $k_1, \ldots, k_b$ constituents of sizes $s_1, \ldots, s_b$, that filter sets $\mathcal{G}_1, \ldots, \mathcal{G}_b$ out of $\mathcal{F}$. Then the graph parameter q defined by*

$$q(G) = p(G \otimes \mathbf{F}_1 \otimes \ldots \otimes \mathbf{F}_b)$$

satisfies $q(\cdot) = \sum_{F \in \mathcal{G}} \alpha_F \hom(F, \cdot)$ with $\mathcal{G} = \bigcap_{i=1}^{b} \mathcal{G}_i$ and can be evaluated on n-vertex graphs with $\prod_{i=1}^{b} k_i$ oracle calls to p on graphs with $n \prod_{i=1}^{b} s_i$ vertices.

In our applications of this corollary, we will have $b \leq 2$. We remark that the idea of implementing desirable functionalities by quantum graphs was used before by Lovasz [34] in the context of *contractors* and *connectors*, by Xia and Curticapean [17, 13] when studying the complexity of the permanent with *combined matchgates*, and by Dvorák [25] to characterize the expressiveness of homomorphism counts.

Proving the known Theorem 1.1 via filters As a warm-up, let us prove the known Theorem 1.1 by a combination of different filters. In the following, let p be a graph motif parameter. First, we invoke a cardinality filter to filter out graphs with exactly $k \in \mathbb{N}$ vertices from the hom-expansion of p. The existence of such filters is shown in Section 4.

Fact 1.4. *Given integers $k, s \in \mathbb{N}$, there is a quantum graph with $s + 1$ constituents, each on $\leq s$ vertices, that filters the graphs with exactly k vertices out of the graphs with $\leq s$ vertices. Its constituents and coefficients can be computed in polynomial time.*

Another straightforward construction allows us to filter S-colored graphs out of all graphs: Given a colorful graph S, a graph H is *S-colored* if there is a homomorphism from H to S. Any colorful graph S itself filters the S-colored graphs out of all graphs. Less straightforwardly, we will also filter for graphs that contain all edge-colors induced by the vertex-coloring of S:

Definition 1.4. *For a colorful graph S, an S-colored graph H is* surjectively S-colored *if, for every edge $ij \in E(S)$, there is at least one edge between vertices of colors i and j in H. We write $E_{ij}(H)$ for the set of edges between vertices of colors i and j in H and call $E_{ij}(H)$ an* edge-color class *of H.*

Besides S itself, any graph obtained by repeatedly splitting off edges from S is surjectively S-colored. That is, we may repeatedly remove any edge $uv \in E_{ij}(S)$ for colors i, j and add an edge between fresh vertices u^* and v^* of colors i and j, to obtain a graph that remains surjectively S-colored. This process can even be repeated until a vertex-colored matching on $2|E(S)|$ vertices is left. Note however that splitting off edges increases the number of vertices. Among surjectively S-colored graphs with exactly $|V(S)|$ vertices, only S itself comes into consideration:

Fact 1.5. *Let S be a colorful graph without isolated vertices. Any surjectively S-colored graph H with $|V(S)| = |V(H)|$ satisfies $S \simeq H$.*

This explains the need for the cardinality filters from Fact 1.4: Combined with a filter for surjectively S-colored graphs, a cardinality filter allows us to filter for S alone. Now, to implement a filter for surjectively S-colored graphs, we can use the inclusion-exclusion principle.

Fact 1.6. *For every colorful graph S, the quantum graph $\mathbf{I} = \sum_{F \subseteq S} (-1)^{|E(S)|-|E(F)|} F$, with F ranging over the edge-subgraphs of S, filters the surjectively S-colored graphs out of the set of all graphs.*

Proof. If H is surjectively S-colored, then $\hom(H, \mathbf{I}) = 1$, since H has exactly one homomorphism to S and none to proper subgraphs. If H is not surjectively S-colored, then $\hom(H, \mathbf{I}) = 0$: Fix any edge $ij \in E(S)$ with $E_{ij}(H) = \emptyset$. Any homomorphism from H to a graph $F \subseteq S$ with $ij \notin E(F)$ is also a homomorphism from H to $F + ij$, and vice versa. Since F and $F + ij$ have opposite signs in $\mathbf{I}$, their contributions to $\hom(H, \mathbf{I})$ cancel out, so we obtain $\hom(H, \mathbf{I}) = 0$ overall. ∎

Finally, we combine the filters to give a proof of Theorem 1.1: Let S be a colorful graph, let G be an input graph and let p be a graph motif parameter that contains S and only graphs with $\leq s$ vertices in its hom-expansion. Using the filters for cardinality (Fact 1.4) and surjectively S-colored graphs (Fact 1.6), we obtain via Corollary 2 a graph motif parameter q that can be evaluated with $2^{|E(S)|}\mathrm{poly}(s)$ oracle calls. By Fact 1.5, we have $q(\cdot) = \hom(S, \cdot)$, so we can compute $\hom(S, G)$ on n-vertex graphs G in time $2^{|E(S)|}\mathrm{poly}(n, s)$ with an oracle for p. By a simple reduction that is described in Section 4, the above procedure can be performed even when p is defined on uncolored graphs and contains the uncolored version S° instead of S itself in its hom-expansion.

New result: Efficient color-surjectivity filters from CFI graphs The main technical contribution of this paper is a new construction of filters for surjectively colored graphs. These new filters are based on graphs that were introduced by Cai, Fürer and Immerman [6] about 30 years ago to show limitations of the *Weisfeiler-Leman method* [45, 44], a particular heuristic for graph isomorphism. The simplest instantiation of this method is the so-called *color refinement* algorithm, which compares the iterated degree sequences of two graphs. This sequence is identical for isomorphic graphs, but already very simple non-isomorphic graphs may have identical iterated degree sequences. The *k-dimensional* Weisfeiler-Leman (k-WL) algorithms for $k \in \mathbb{N}$ are higher-dimensional variants of color refinement that compare iterated neighborhoods of k-tuples of vertices. These algorithms have also been linked to homomorphism counts [20, 25] and were investigated in group-theoretic, algebraic and category-theoretic settings [3, 1, 22].

Cai, Fürer and Immerman [6] showed that, for every $k \in \mathbb{N}$, there exist two non-isomorphic graphs that are not distinguished by k-WL. It follows that no fixed level in the hierarchy of k-WL algorithms can solve the graph isomorphism problem. By a known modification of the original CFI construction [38, 36], we can transform any connected colorful graph S into two non-isomorphic S-colored graphs $X = X(S)$ and $\tilde{X} = \tilde{X}(S)$, as shown in Figure 1. These graphs each have at most $2^{\Delta(S)-1}$ vertices per color class and come with a remarkable property: While being non-isomorphic, deleting all edges between colors i and j for *any* edge $ij \in E(S)$ renders X and $\tilde{X}$ isomorphic. Intuitively speaking, this means that X and $\tilde{X}$ can only be distinguished by "inspecting" all of their edge-color classes. Thus, if some edge-color class of S is missing in a graph H, then homomorphisms from H cannot distinguish X and $\tilde{X}$, since they cannot "inspect" this missing edge-color class. It follows

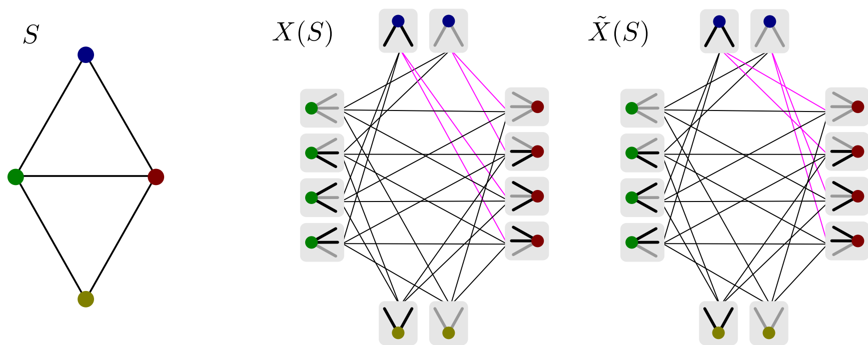

Figure 1: A colorful graph S and the CFI graphs $X(S)$ and $\tilde{X}(S)$. As indicated in the figure, for every $i \in V(S)$, the i-colored vertices in $X(S)$ and $\tilde{X}(S)$ represent even assignments in $\{0,1\}^{I(i)}$. For every edge $ij \in E(S)$, an edge is present between two assignments in $X(S)$ if they agree on their value for ij. The same is true for $\tilde{X}(S)$ except that the pink edge-color class consisting of the edges between blue and red vertices is complemented. The exceptional automorphism group of $\tilde{X}(S)$ implies that the choice of edge-color class to be complemented is *irrelevant*: For all $ij \in E(S)$, complementing $E_{ij}(X(S))$ yields the same graph $\tilde{X}(S)$ modulo isomorphism.

that $\hom(H,X) = \hom(H,\tilde{X})$ and thus $\hom(H, X - \tilde{X}) = 0$.[4] For $H = S$ however, it can be shown that $\hom(S, X - \tilde{X}) = 2^{|E(S)|-|V(S)|+1}$. Normalizing by this number, we obtain:

LEMMA 1.1. *Given a colorful graph S that is connected and has no isolated vertices, there exists a quantum graph $\mathbf{X}$ with two constituents, each with at most $2^{\Delta(S)-1}$ vertices per color class, that filters $\{S\}$ out of $\{S\} \cup \mathcal{N}(S)$, where $\mathcal{N}(S)$ is the set of graphs that are not surjectively S-colored. The constituents and coefficients of $\mathbf{X}$ can be computed in time $O(4^{\Delta(S)}|V(S)|)$.*

Note that $\mathbf{X}$ is technically not a filter for surjectively S-colored graphs, as it only filters $\{S\}$ out of $\{S\} \cup \mathcal{N}(S)$. We will verify however that this suffices to prove Theorem 1.4 along the lines of Facts 1.4 and 1.6, replacing the inclusion-exclusion filter from Fact 1.6 by our new CFI filters from Lemma 1.1. Then, rather than computing a graph product with a quantum graph on $2^{|E(S)|}$ constituents, each on $|V(S)|$ vertices, it suffices to compute a graph product with a quantum graph with two constituents, each on $2^{\Delta(S)-1}|V(S)|$ vertices.

Finally, in order to obtain the full Theorem 1.4, we need to reduce from $\hom(S, \cdot)$ for colorful S to graph motif parameters p with $T^\circ \triangleleft p$ for some supergraph $T \supseteq S$. This requires adding "dummy edges", which yields technicalities that lead to requirements (a)–(c) in the theorem to avoid unwanted cancellations.

1.5 Organization of the paper The general proof outline for Theorem 1.4 was already given in the introduction. The remainder of the paper contains five sections: Section 2 collects preliminaries on graph width measures and known hardness results for counting homomorphisms from colorful graphs. In Section 3, we describe the CFI graph construction and prove Lemma 1.1. The full proof of Theorem 1.4 is given in Section 4, and the theorem is used in Section 5 to prove Theorems 1.5–1.7. In Section 6, we conclude with an outlook for future work based on this paper.

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2 Preliminaries

In this paper, a *counting problem* is a function $g : \{0,1\}^* \to \mathbb{Q}$. The counting problem #SAT asks, on input a binary encoding of a Boolean formula φ, to count the satisfying assignments for φ. We say that a counting problem g is #P*-hard* if there is a polynomial-time Turing reduction from #SAT to g, i.e., a polynomial-time algorithm that counts the satisfying assignments of an input formula φ when given oracle access to g. As shown by Valiant [43], the problem of counting the perfect matchings in an input graph G is #P-hard.

[4] Similar features of CFI graphs were observed by Roberson [38] in the study of uncolored *oddomorphisms*.

Graphs Graphs G will be undirected and may feature self-loops but no parallel edges. As described in the introduction, they may be vertex-colored. Given a vertex $v \in V(G)$, we write $I(v)$ for the set of edges incident with v, assuming that G is unique from the context. The maximum degree of G will be denoted by $\Delta(G)$. The treewidth $\mathrm{tw}(G)$ of G is a measure of "tree-likeness" that we do not need to define formally in this paper. For a definition, see [18, Chapter 7].

An *elementary* $r \times r$ *wall* for $r \in \mathbb{N}$ is the subgraph obtained from an $r \times r$ square grid by deleting every odd-indexed edge from every odd-indexed column and every even-indexed edge from every even-indexed column. An $r \times r$ *wall* is any graph obtained by subdividing some edges of an elementary $r \times r$ wall; note that its maximum degree is bounded by 3. The *wall order* $\mathrm{wall}(G)$ of a graph G is the maximum $r \in \mathbb{N}$ such that G contains an $r \times r$ wall subgraph. By a highly nontrivial result due to Chekuri and Chuzhoy [7], the *treewidth* $\mathrm{tw}(G)$ of G is polynomially equivalent to the largest number $s \in \mathbb{N}$ such that G contains a $s \times s$ square grid minor, and that number s in turn is linearly equivalent to the wall order.

THEOREM 2.1. ([7]) *For any graph G of treewidth $t \in \mathbb{N}$, we have $\mathrm{wall}(G) \in \Omega(t^{1/100})$ and $\mathrm{wall}(G) \in O(t)$. Moreover, there is a randomized polynomial-time algorithm that, given as input a graph G of treewidth t, outputs an $r \times r$ wall subgraph of G with $r \in \Omega(t^{1/100})$.*

It is known that $\mathrm{wall}(G) \in \Omega(t^{1/10})$, but this stronger bound does not come with an algorithm that constructs the relevant wall subdivisions in graphs of large treewidth [12].

Hardness of colorful homomorphisms Given a class $\mathcal{H}$ of colorful graphs, let $\#\mathrm{ColHom}(\mathcal{H})$ be the problem of computing the number of color-preserving homomorphisms $\mathrm{hom}(H, G)$ on input graphs $H \in \mathcal{H}$ and G. For recursively enumerable graph classes $\mathcal{H}$ of unbounded treewidth, the problem $\#\mathrm{ColHom}(\mathcal{H})$ is known to be #W[1]-hard on parameter $|V(H)|$, as shown, e.g., by Curticapean and Marx [16], who called the problem $\#\mathrm{PartitionedSub}(\mathcal{H})$. That problem asks to count color-preserving subgraph copies, but for colorful graphs H, these correspond bijectively to color-preserving homomorphisms.

The #W[1]-hardness of $\#\mathrm{ColHom}(\mathcal{H})$ was shown in two steps, which can be adapted to a #P-hardness proof. First, let $\mathcal{G}$ denote the class of colorful square grids. The proof of Theorem 3.2 in [16] shows #W[1]-hardness of $\#\mathrm{ColHom}(\mathcal{G})$ by a parameterized reduction from counting k-cliques. Counting k-cliques is actually #P-hard, as the standard NP-hardness reduction from the Boolean satisfiability problem preserves the number of solutions. The straightforward reduction from counting k-cliques to $\#\mathrm{ColHom}(\mathcal{G})$ in the proof of Theorem 3.2 in [16] runs in polynomial time, so we obtain:

THEOREM 2.2. *The problem $\#\mathrm{ColHom}(\mathcal{G})$ for the class $\mathcal{G}$ of colorful square grids is #P-hard.*

By Theorem 2.1, graphs of large treewidth contain large square grids as minors. This can be used to reduce $\#\mathrm{ColHom}(\mathcal{G})$ to $\#\mathrm{ColHom}(\mathcal{H})$ for large-treewidth classes $\mathcal{H}$: A *minor model* of a graph A in a graph B is a family of disjoint connected subsets $S_v \subseteq V(B)$ for each $v \in V(A)$ such that, for each edge $uv \in E(A)$, at least one edge runs between S_u and S_v.

LEMMA 2.1. (SEE PROOF OF LEMMA 3.1 IN [16]) *There is a polynomial-time algorithm to transform an input graph G and a minor model of a colorful graph A in a colorful graph B into a graph G' with $|V(G)| \cdot |V(B)|$ vertices such that $\mathrm{hom}(A, G) = \mathrm{hom}(B, G')$.*

We remark that proving hardness for the colorful version of counting homomorphisms is somewhat easier than for the uncolored problem. As discussed in the introduction, the previously known techniques to remove colors require exponential running time in H.

3 Constructing color-surjectivity filters

The following constructions are essentially due to Cai, Fürer and Immerman [6], up to modifications that also already appeared in previous works [38, 36]. See Figure 1 for an overview. The graphs constructed in this section are obtained from an arbitrary connected and colorful "base graph" S by defining two constraint satisfaction problems (CSPs) from S and then encoding these CSPs as graphs. For our purposes, we fix a colorful graph S and consider a *CSP* to be a pair $\Gamma = (\{A_v\}_{v \in V(S)}, \{R_{uv}\}_{uv \in E(S)})$ consisting of

- a set of assignments $A_v \subseteq \{0,1\}^{I(v)}$ for each $v \in V(S)$, and

- a relation $R_{uv} \subseteq A_u \times A_v$ for each $uv \in E(S)$.

A *solution* of Γ is a family of assignments $\{a_v\}_{v\in V(S)}$ such that $(a_u, a_v) \in R_{uv}$ for all $uv \in E(S)$. Slightly abusing notation, we will also view Γ as an S-colored graph with $V_v(\Gamma) = A_v$ for $v \in V(S)$ and $E_{uv}(\Gamma) = R_{uv}$ for $uv \in E(S)$, and we ignore edge-directions in the relations. The following is immediate from the definition of homomorphisms.

Fact 3.1. *The number of solutions to* Γ *equals* $\hom(S, \Gamma)$.

CFI graphs are constructed from particular CSPs in which A_v for $v \in V(S)$ contains the *even* assignments in $\{0,1\}^{I(v)}$, i.e., those assignments with an even numbers of 1s. Given a "charge function" $c : E(S) \to \{0,1\}$, we define $\Gamma(S, c)$ by setting, for all $uv \in E(S)$,

$$R^c_{uv} = \{(a_u, a_v) \in A_u \times A_v \mid a_u(uv) \equiv_2 a_v(uv) + c(uv)\},$$

with equivalence taken modulo 2. In other words, if $c(uv) = 0$, then R^c_{uv} requires assignments a_u and a_v to agree on their value for uv. If $c(uv) = 1$, then R^c_{uv} requires them to disagree on this value. Given a set of edges $F \subseteq E(S)$, let us introduce the notation $\chi_F \in \{0,1\}^{E(S)}$ to describe the assignment that is 1 on F and 0 otherwise. We also abbreviate χ_F as χ_e and $\chi_{e,e'}$ when $|F| \leq 2$. Then we choose an arbitrary edge $e^* \in E(S)$ and define the quantum graph

$$\mathbf{X}(S) = \frac{\Gamma(S, \chi_\emptyset) - \Gamma(S, \chi_{e^*})}{2^{|E(S)|-|V(S)|+1}}. \tag{3.2}$$

The CSP interpretation allows us to determine the number of homomorphisms into $\mathbf{X}(S)$ explicitly.

Lemma 3.1. *For any connected and colorful graph* S*, we have* $\hom(S, \mathbf{X}(S)) = 1$.

Proof. By Fact 3.1, the value $\hom(S, \mathbf{X}(S))$ is the difference of the solution counts for the two constituents $\Gamma(S, \chi_\emptyset)$ and $\Gamma(S, \chi_{e^*})$, up to normalization by $2^{|E(S)|-|V(S)|+1}$. We determine these solution counts:

$\Gamma(S, \chi_\emptyset)$: The solutions for this CSP correspond bijectively to edge-subsets $B \subseteq E(S)$ such that each vertex $v \in V(S)$ has even degree in B. As S is connected, there are precisely $2^{|E(S)|-|V(S)|+1}$ such edge-subsets, as these form the *cycle space* of S, which is a subspace of $\mathbb{Z}_2^{E(S)}$ of dimension $|E(S)|-|V(S)|+1$.

$\Gamma(S, \chi_{e^*})$: We show that this CSP has no solutions. Choose u arbitrarily as one of the endpoints of e^*. The solutions for $\Gamma(S, \chi_{e^*})$ correspond bijectively to edge-subsets $B \subseteq E(S)$ such that each vertex $v \in V(S)$ has even degree in B, except for u, which has odd degree. By the graph handshaking lemma, no such B exists, as it would induce a graph with precisely one vertex of odd degree.

Combining the above, the lemma follows from the definition of $\mathbf{X}(S)$. ◻

In the following lemma, we establish an extremely useful isomorphism property of the above CSPs and the resulting CFI graphs; we view two CSPs as isomorphic if they are isomorphic as colored graphs: Given a charge function $c \in \{0,1\}^{E(S)}$ and two incident edges $vu, vw \in E(S)$, we can flip c at vu and vw and obtain a new charge function c' that produces an isomorphic CFI graph.

More formally, given functions c and c' with range $\{0,1\}$, let $c \oplus c'$ be their point-wise sum modulo 2. Slightly abusing notation, we also write $c \oplus c'$ when the domains D and D' of c and c' disagree and then define the domain of $c \oplus c'$ as $D \cap D'$.

Lemma 3.2. *For all* $c \in \{0,1\}^{E(S)}$ *and* $vu, vw \in E(S)$*, we have* $\Gamma(S, c) \simeq \Gamma(S, c \oplus \chi_{vu,vw})$.

Proof. We define an isomorphism f from $\Gamma(S, c)$ to $\Gamma(S, c \oplus \chi_{vu,vw})$ by setting

$$f(a) = \begin{cases} a \oplus \chi_{vu,vw} & a \in A_v, \\ a & a \in A_z \text{ with } z \neq v. \end{cases}$$

As any $a \in A_z$ for $z \in V(S)$ is even and its image $f(a)$ is obtained by flipping two (if $z = v$) or zero (if $z \neq v$) entries, this image is also even and thus contained in A_z. The function f is therefore well-defined.

To check that f preserves the edge-relation, let us abbreviate $c' = c \oplus \chi_{vu,vw}$ and $a' = f(a)$ for all $a \in V(\Gamma)$. Then, for $z \in \{v, w\}$, we have $(a_u, a_z) \in R^c_{uz}$ iff $(a'_u, a'_z) \in R^{c'}_{uz}$. This is because $a_u(uz) = a_z(uz)$ iff $a'_u(uz) \neq a'_z(uz)$, while all other relations are trivially preserved. This shows that f is indeed an isomorphism from $\Gamma(S, c)$ to $\Gamma(S, c \oplus \chi_{vu,vw})$. ∎

Using this, we can "push charge" from any edge to another edge in the same connected component.

LEMMA 3.3. *Let $e, e' \in E(S)$. If S is connected, then $\Gamma(S, \chi_e) \simeq \Gamma(S, \chi_{e'})$.*

Proof. Let $P = (e, \dots, e')$ be a sequence of edges in S in which consecutive edges share a common endpoint; this sequence exists because S is connected. The required isomorphism follows from repeated applications of Lemma 3.2 along consecutive edges in P. ∎

We can use this lemma to show that every edge-color is relevant in the CFI construction, in the sense that the constituents of $\mathbf{X}(S)$ become isomorphic when any edge-color class is deleted. Given an S-colored graph F and a proper subgraph $S' \subsetneq S$ of the colorful graph S, note that the tensor product $F \otimes S'$ is canonically isomorphic to the graph obtained from F by deleting the edge-color classes corresponding to edges in $E(S) \setminus E(S')$. Mildly abusing notation, we view $F \otimes S'$ as an actual subgraph of F.

COROLLARY 3. *For any proper subgraph S' of S, we have $\Gamma(S, \chi_{e^*}) \otimes S' \simeq \Gamma(S, \chi_\emptyset) \otimes S'$.*

Proof. Let $e \in E(S) \setminus E(S')$ with $e = uv$. By Lemma 3.3, we can "push" the charge from e^* to e, i.e., we have $\Gamma(S, \chi_{e^*}) \simeq \Gamma(S, \chi_e)$. This also implies $\Gamma(S, \chi_{e^*}) \otimes S' \simeq \Gamma(S, \chi_e) \otimes S'$, because deleting entire vertex-color and edge-color classes from two isomorphic colored graphs preserves their isomorphism. On the other hand, we have $\Gamma(S, \chi_e) \otimes S' \simeq \Gamma(S, \chi_\emptyset) \otimes S'$ by the trivial identity isomorphism, because $\Gamma(S, \chi_e)$ and $\Gamma(S, \chi_\emptyset)$ differ only on edges in R_{uv}. Composing the last two isomorphisms yields the claim. ∎

Finally, we collect the facts shown in this section to prove Lemma 1.1.

Proof. [Proof of Lemma 1.1] Given a connected graph S, define the quantum graph $\mathbf{X}(S)$ as in (3.2). Viewed as S-colored graphs, the CSPs $\Gamma(S, \chi_\emptyset)$ and $\Gamma(S, \chi_{e^*})$ satisfy the requirements on the color class sizes and can be computed in the required time. Note that up to $O(4^{\Delta(S)})$ time may be required to write down all edges in a given edge-color class of the two graphs.

Lemma 3.1 shows that $\hom(S, \mathbf{X}) = 1$. To prove that $\hom(H, \mathbf{X}) = 0$ for graphs H that are not surjectively S-colored, let H be such a graph. We may assume that H is S-colored, as otherwise $\hom(H, \mathbf{X}) = 0$ because the constituents of $\mathbf{X}$ are S-colored. It follows that H is S'-colored for $S' \subsetneq S$, and we obtain

$$\hom(H, \mathbf{X}(S)) = \hom(H, \mathbf{X}(S) \otimes S') = 0,$$

where the first equality holds because the image of H in any constituent X of $\mathbf{X}(S)$ is contained in $X \otimes S'$, and the second equality holds because the two constituents of $\mathbf{X}(S) \otimes S'$ are isomorphic by Corollary 3. This proves the lemma. ∎

4 Filtering a linear combination

In this section, we prove Theorem 1.4 along the outline described in Section 1.4 of the introduction: Given a set $\mathcal{H}$ of pairwise non-isomorphic uncolored graphs and access to a graph parameter

$$p(\cdot) = \sum_{H \in \mathcal{H}} \alpha_H \hom(H, \cdot),$$

we wish to determine $\hom(S, G)$ for a colorful graph S and a colored graph G.

First, we need to address the conversion from colored to uncolored graphs: Recall that we write G° for the uncolored graph obtained from G by ignoring colors. We write $\mathbf{G}^\circ$ for the quantum graph obtained by applying this operator constituent-wise. Moreover, we write H^c for the colored graph obtained from H by applying c as coloring.

LEMMA 4.1. *Given a set of colors C, let p_{col} denote the graph motif parameter on colored graphs obtained from the uncolored graph parameter p via*

$$p_{\mathrm{col}}(\cdot) = \sum_{H \in \mathcal{H}} \alpha_H \sum_{c : V(H) \to C} \hom(H^c, \cdot). \tag{4.3}$$

Given a colored graph G, we have $p(G^\circ) = p_{\mathrm{col}}(G)$. After collecting for isomorphic terms, the coefficient of any colorful graph F with $|C|$ vertices in p_{col} equals $\alpha_{F^\circ} |\mathrm{aut}(F^\circ)|$.

Proof. We first show $p(G^\circ) = p_{\mathrm{col}}(G)$ when $p(\cdot) = \hom(H, \cdot)$ for a fixed uncolored graph H. The general claim follows by linearity. Given a graph G on colors C, we have $\hom(H, G^\circ) = \sum_{c:V(H)\to C} \hom(H^c, G)$, because any homomorphism f from H into G° induces a unique coloring c_f of H: For $v \in V(H)$, let $c_f(v)$ be the color of $f(v)$ in G. The set of homomorphisms from H to G° can be partitioned according to these colorings, and the homomorphisms inducing a given coloring $c : V(H) \to C$ are precisely the homomorphisms from H^c to G.

For the second statement, let F be a colorful graph with some fixed coloring b. The terms for graphs isomorphic to F correspond to colorings c such that $F \simeq (F^\circ)^c$. For any such coloring, the isomorphism from F to $(F^\circ)^c$ is uniquely determined and induces a unique automorphism of F°. Moreover, any automorphism of F° induces such a coloring c. ∎

Our proof relies on a general version of the cardinality filter discussed in the introduction.

DEFINITION 4.1. *Consider a partition of a color set C into $r \in \mathbb{N}$ parts $C_1, \ldots, C_r$ that are annotated with numbers $k_1, \ldots, k_r \in \mathbb{N}$. We call $\eta = (C_1, \ldots, C_r, k_1, \ldots, k_r)$ a* color-coarsening *of C with r parts and say that H is η-*coarsened *if $\sum_{j \in C_i} |V_j(H)| = k_i$ holds for all $i \in [r]$.*

That is, if H is η-coarsened, then for every $i \in [r]$, there are exactly k_i vertices in H with colors from C_i. In particular, a graph H on color set C has $k \in \mathbb{N}$ vertices iff it is (C, k)-coarsened.

LEMMA 4.2. *Let η be a color-coarsening, with $r \in \mathbb{N}$ parts, of a color set C. For all $s \in \mathbb{N}$, there exists a quantum graph $\mathbf{N}$ that filters the η-coarsened graphs out of the graphs with $\leq s$ vertices. The coefficients and constituents of $\mathbf{N}$ can be computed in $s^{O(r)}$ time on input η.*

Proof. Let $\eta = (C_1, \ldots, C_r, k_1, \ldots, k_r)$. Given $a = (a_1, \ldots, a_r) \in [s+1]^r$, define a graph N_a: For $i \in [r]$, this graph contains exactly a_i vertices of color j, for all colors $j \in C_i$. All vertices in N_a have self-loops and edges to all other vertices. Given a colored graph H with colors C and $|V(H)| \leq s$, write $n_i(H) = \sum_{j \in C_i} |V_j(H)|$ for $i \in [r]$. For all $a \in [s+1]^r$, we have

$$\hom(H, N_a) = \prod_{i \in [r]} a_i^{n_i(H)}. \tag{4.4}$$

For any fixed graph H, the right-hand side of the above equation can be viewed as a multivariate polynomial $p_H \in \mathbb{Q}[a_1, \ldots, a_r]$ with maximum degree s, since $|V(H)| \leq s$. This polynomial has exactly one monomial, and this monomial is $a_1^{k_1} \ldots a_r^{k_r}$ iff H is η-coarsened. Thus, the coefficient $c_{k_1,\ldots,k_r}$ of $a_1^{k_1} \ldots a_r^{k_r}$ in p_H is 1 if H is η-coarsened, and it is 0 otherwise. By polynomial interpolation, any fixed coefficient of a multivariate polynomial can be expressed as a linear combination of evaluations of the polynomial on a sufficiently large grid. Thus, there are coefficients β_a for $a \in [s+1]^r$ such that

$$c_{k_1,\ldots,k_r} = \sum_{a \in [s+1]^r} \beta_a p_H(a) = \sum_{a \in [s+1]^r} \beta_a \hom(H, N_a),$$

where the coefficients β_a can be computed in polynomial time by relating the evaluations and coefficients of p_H through a full-rank linear system of equations on $(s+1)^r$ indeterminates. It follows that $\mathbf{N} := \sum_{a \in [s+1]^r} \beta_a N_a$ satisfies the requirements of the lemma. ∎

To prove Theorem 1.4, we invoke Lemma 1.1 from Section 3 together with Lemma 4.2 on $r \leq 2$ and $s = \max_{H \triangleleft p} |V(H)|$. It will not suffice to choose $r = 1$, since we may need to distinguish between "relevant" and "dummy" vertices in parts of the proof.

Proof. [Proof of Theorem 1.4] We may assume that S is connected: If S has connected components $S_1, \ldots, S_q$ with color sets $C_1, \ldots, C_q$ for $q \geq 2$, then we have $\hom(S, G) = \prod_{i=1}^{q} \hom(S_i, G_i)$, where G_i is the restriction of G to vertices of color C_i, so it suffices to compute $\hom(S_i, G_i)$ for $i \in [q]$ independently. Assuming then that S is connected, let $\mathbf{X} = \mathbf{X}(S)$ be the quantum graph from Lemma 1.1. Its two constituents on $\leq 2^{\Delta(S)}|V(S)|$ vertices and coefficients can be computed in $O(4^{\Delta(S)}|V(S)|)$ time.

We first describe the reduction in Case (a) of the theorem. Lemma 4.2 yields a quantum graph $\mathbf{N}$ that filters the $(C, |V(S)|)$-coarsened graphs out of the graphs with $\leq s$ vertices. The $\mathrm{poly}(s)$ constituents and coefficients of $\mathbf{N}$ can be computed in $\mathrm{poly}(s)$ time. The $(C, |V(S)|)$-coarsened graphs are precisely the graphs on color set C with $|V(S)|$ vertices. By Fact 1.5, the surjectively S-colored graphs among those graphs are isomorphic to S. Using Corollary 2 with $\mathbf{X}$ and $\mathbf{N}$ on p_{col}, and using Lemma 4.1, we can thus evaluate $\alpha_{S^\circ}|\mathrm{aut}(S^\circ)| \cdot \hom(S, G)$ with $2^{\Delta(S)}\mathrm{poly}(s)$ oracle calls of the form $p_{\mathrm{col}}(R) = p(R^\circ)$. Since $\hom(S, S) = 1$, we can moreover compute $\alpha_{S^\circ}|\mathrm{aut}(S^\circ)| \neq 0$ by invoking the above steps with $G = S$. This allows us to compute $\hom(S, G)$.

In Case (b), we proceed similarly, but we add "padding" to the host graphs to capture the surplus edges in $T \supseteq S$. That is, for any S-colored graph G, we define a graph G_{pad} as follows, starting from G:

1. For every vertex $v \in V(T) \setminus V(S)$, add a vertex with the same color as v to G_{pad}.

2. For every edge $ij \in E(T) \setminus E(S)$, add all edges uv between colors i and j in G_{pad}.

Note that G_{pad} is T-colored. For any T-colored graph H, consider its restriction $H \otimes S$ to vertex- and edge-colors from S. Any homomorphism from H to G_{pad} can be restricted to a homomorphism from $H \otimes S$ to G by deleting the image vertices and edges not contained in G. Since "padding" color classes in G_{pad} are fully connected, there exists a number $c_H \neq 0$, independent of G, such that any homomorphism f from $H \otimes S$ to G admits exactly c_H homomorphisms from H to G_{pad} that restrict to f. That is,

$$\hom(H, G_{\mathrm{pad}}) = c_H \cdot \hom(H \otimes S, G). \tag{4.5}$$

Let $C_S \subseteq C_T$ denote the colors of S and T. We set $\eta = (C_S, C_T \setminus C_S, |V(S)|, |V(T)| - |V(S)|)$ and write $\mathcal{C}$ for the set of T-colored and η-coarsened graphs H^c with $H \in \mathcal{H}$ and $c : V(H) \to C$ such that $H^c \otimes S$ is surjectively S-colored. Note that $\mathcal{C}$ may contain pairs of isomorphic graphs. In fact, since $|V(H \otimes S)| = |V(S)|$ for all $H \in \mathcal{C}$, it follows by Fact 1.5 that

$$H \otimes S \simeq S \quad \text{for all } H \in \mathcal{C}. \tag{4.6}$$

Using Corollary 2 with $\mathbf{X}$ and $\mathbf{N}$ on p_{col}, we can thus use $2^{\Delta(S)}\mathrm{poly}(s)$ oracle calls to evaluate

$$\sum_{H \in \mathcal{C}} \alpha_{H^\circ} \hom(H, G_{\mathrm{pad}}) \;=\; \sum_{H \in \mathcal{C}} c_H \alpha_{H^\circ} \hom(H \otimes S, G) \;=\; \sum_{H \in \mathcal{C}} c_H \alpha_{H^\circ} \hom(S, G),$$

where the equalities are due to (4.5) and (4.6). Note that the last sum ranges over $H \in \mathcal{C}$ but only involves homomorphism counts from S. Since all graphs $H \in \mathcal{C}$ have exactly $|V(T)|$ vertices, their coefficients α_{H° all have the same signs by the assumptions of Case (b), so it follows that $q_{\mathcal{C}} := \sum_{H \in \mathcal{C}} c_H \alpha_{H^\circ} \neq 0$. As in Case (a), we can then determine $q_{\mathcal{C}}$ with additional oracle calls by setting $G = S$ and finally compute $\hom(S, G)$.

Case (c) is shown similarly to Case (b): We construct G_{pad} and define $\mathcal{C}$, and then we evaluate $\sum_{H \in \mathcal{C}} c_H \alpha_{H^\circ} \hom(S, G)$ as above. All graphs in $\mathcal{C}$ are T-colored and have exactly $|V(T)|$ vertices, so they are edge-subgraphs of T. By the assumptions of Case (c), this only leaves recolored versions of T in $\mathcal{C}$, and thus $q_{\mathcal{C}} = \sum_{H \in \mathcal{C}} c_H \alpha_{H^\circ} \neq 0$ and we can proceed as in the last steps of Case (b) to compute $\hom(S, G)$. ∎

5 Applications of the main reduction

In this section, we show Theorems 1.5 and 1.6 as applications of Theorem 1.4. We start with the result for $\#\mathrm{Hom}(\mathcal{H})$, the problem of counting uncolored homomorphism counts from a fixed class of patterns $\mathcal{H}$. Our new theorem translates a fundamental #W[1]-hardness result in parameterized complexity by Dalmau and Jonsson [19] to a #P-hardness result.

Proof. [Proof of Theorem 1.5] We show #P-hardness of $\#\mathrm{Hom}(\mathcal{H})$ by reduction from $\#\mathrm{ColHom}(\mathcal{G})$ for the class of colorful grids, which is #P-hard by Theorem 2.2. As input for $\#\mathrm{ColHom}(\mathcal{G})$, let $H \in \mathcal{G}$ be a colorful grid and let G be a colored graph. Using the assumption of the theorem, we can compute a colorful graph T with $T^\circ \in \mathcal{H}$ in time $\text{poly}(|V(H)|)$ such that $S \subseteq T$ for a wall S that is large enough to contain H as a minor. We either use Theorem 2.1 to find S in randomized polynomial time or use the deterministic algorithm to find it.

Using Lemma 2.1, we then compute a graph G' with $\hom(H, G) = \hom(S, G')$ in polynomial time. Using Theorem 1.4, we determine $\hom(S, G')$ in polynomial time with an oracle for $\hom(T^\circ, \cdot)$: Indeed, we have $\Delta(S) \leq 3$ and T° is the only graph appearing in the hom-expansion of $\hom(T^\circ, \cdot)$, so the sign condition in Case (b) in Theorem 1.4 is trivially fulfilled, yielding the desired reduction to $\#\text{Hom}(\mathcal{H})$. ∎

We continue with the subgraph counting problems $\#\text{Sub}(\mathcal{H})$. Significant technical effort and methods specific to parameterized complexity were required to obtain #W[1]-hardness for $\#\text{Sub}(\mathcal{H})$ in the original paper by Marx and Curticapean [16]. Even the machinery developed in follow-up works [15] did not suggest that a general #P-hardness result was in reach.

Proof. [Proof of Theorem 1.6] By Theorem 2.2 and Lemma 2.1, the problem $\#\text{ColHom}(\mathcal{W})$ for the class of colorful walls is #P-hard. We reduce it to $\#\text{Sub}(\mathcal{H})$. Let $S \in \mathcal{W}$ and a graph G be given as input for $\#\text{ColHom}(\mathcal{W})$. Using the assumption of the theorem, we find a graph $H \in \mathcal{H}$ and a matching M in H with $|E(S)|$ edges in polynomial time. By identifying the endpoints of edges in M, any graph on $|E(S)|$ edges can be found in polynomial time as a subgraph of some quotient of H. In particular, there is a colorful graph $T \supseteq S$ such that T° is a quotient of H. This graph T and the subgraph copy of S in T can be computed in polynomial time.

We use Theorem 1.4 to compute $\hom(S, G)$ in polynomial time with an oracle for $\text{sub}(H, \cdot)$: Note that $\Delta(S) \leq 3$ and that all graphs $F \triangleleft \text{sub}(H, \cdot)$ have at most $|V(H)|$ vertices. By Fact 1.1, we have $T^\circ \triangleleft \text{sub}(H, \cdot)$, since T° is a quotient of H, and all graphs R with $|V(R)| = |V(T)|$ have the same sign in the hom-expansion. Invoking Theorem 1.4 with Case (b) yields the desired reduction from $\#\text{ColHom}(\mathcal{W})$ to $\#\text{Sub}(\mathcal{H})$. ∎

We conclude with a proof for the induced subgraph counting problems $\#\text{Ind}(\mathcal{H})$.

Proof. [Proof of Theorem 1.7] For all $t \in \mathbb{N}$, we can compute a graph H_t with $|V(H_t)| \geq t$ in polynomial time. At least one of H_t or its complement has average degree $\Omega(t)$, and we may assume that H_t has, as we could otherwise complement H_t and the host graph without changing the induced subgraph count. Using known polynomial-time algorithms [2, Fact 7], we can then find a minor model of K_q in H_t for $q \in \Omega(t/\sqrt{\log t})$. From this minor model, we can then find a $q' \times q'$ wall subgraph S with $q' \in \Omega(t^{1/2-\epsilon})$ in H_t.

By Theorem 2.2 and Lemma 2.1, the problem $\#\text{ColHom}(\mathcal{E})$ for the class $\mathcal{E}$ of colorful elementary walls is #P-hard. Let $W \in \mathcal{E}$ and let G be a graph. As in the previous paragraph, we can find a graph $H_t \in \mathcal{H}$ such that $S^\circ \subseteq H_t$ for a subdivision S of W. Lemma 2.1 allows us to construct a graph G' such that $\hom(W, G) = \hom(S, G')$. By Fact 1.2, the graphs $F \triangleleft \text{ind}(H_t, \cdot)$ have at most $|V(H_t)|$ vertices, and among those F in the hom-expansion with $|V(F)| = |V(H_t)|$, the graph H_t itself is edge-minimal. We can therefore invoke Theorem 1.4 in Case (c) to compute $\hom(W', G')$ in polynomial time with an oracle for $\text{ind}(H_t, \cdot)$. ∎

6 Conclusion and future work

We proved #P-hardness for certain families of graph parameters $(p_t)_{t\in\mathbb{N}}$ in which each parameter p is a linear combination $p(\cdot) = \sum_F \alpha_F \hom(F, \cdot)$ with finitely many terms, and the expansions contain polynomial-time constructible graphs of polynomial treewidth. To this end, we used tensor products with quantum graphs to narrow down the hom-expansion of p to $\hom(S, \cdot)$ for some high-treewidth graph S. Combined with known combinatorial results that characterize the graphs $S \triangleleft p$, this reduction yields #P-hardness results for counting homomorphisms, subgraphs, and induced subgraphs from polynomial-time constructible pattern classes.

There is potential for interesting follow-up works: Most importantly, it is likely that the requirements in Theorem 1.4 can be weakened. How close can we get to merely requiring $T^\circ \triangleleft p$ for some supergraph $T \supseteq S$, without imposing additional requirements on the coefficients of other graphs besides T°? It may not even be necessary to provide T as an input, as it may be possible to find T using the oracle for p.

Other possible follow-up works include strengthening the #P-hardness results to lower bounds under the exponential-time hypothesis #ETH. Under certain circumstances, it may also be possible to remove cardinality filters from the main proof. This would yield reductions that require only two oracle calls, which may render them applicable for $\mathsf{C_=P}$-completeness results [14, 27]. Moreover, removing cardinality filters may also make the techniques applicable to modular counting problems: While the coefficients of cardinality filters introduce uncontrollable divisions, the CFI-based filters for surjectively S-colored graphs only require divisions by powers of 2.

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