

On the I/O complexity of the k -nearest neighbor problem

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Abstract

We consider static, external memory indexes for exact and approximate versions of the k -nearest neighbor (k -NN) problem, and show new lower bounds under a standard *indivisibility* assumption:

- Polynomial space indexing schemes for high-dimensional k -NN in Hamming space cannot take advantage of block transfers: $\Omega(k)$ block reads are needed to answer a query.
- For the ℓ_∞ metric the lower bound holds even if we allow c -approximate nearest neighbors to be returned, for $c \in (1, 3)$.
- The restriction to $c < 3$ is necessary: For every metric there exists an indexing scheme in the *indexability model* of Hellerstein et al. using space $O(kn)$, where n is the number of points, that can retrieve k c -approximate nearest neighbors using optimal $\lceil k/B \rceil$ I/Os, where B is the block size.
- For specific metrics, data structures with better approximation factors are possible. For k -NN in Hamming space and every

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approximation factor $c > 1$ there exists a polynomial space data structure that returns k c -approximate nearest neighbors in $\lceil k/B \rceil$ I/Os.

To show these lower bounds we develop two new techniques: First, to handle that approximation algorithms have more freedom in deciding which result set to return we develop a *relaxed* version of the λ -set workload technique of Hellerstein et al. This technique allows us to show lower bounds that hold in $d \geq n$ dimensions. To extend the lower bounds down to $d = O(k \log(n/k))$ dimensions, we develop a new deterministic dimension reduction technique that may be of independent interest.

1 Introduction

The DEEP1B data set [8] is among the largest image data sets that has been examined in the similarity search literature. From each of $n = 10^9$ images, a 96-dimensional vector has been extracted from an intermediate layer of a pre-trained deep neural network, a state-of-the-art method for semantically meaningful feature vectors [9]. Such feature vectors can be thought of as compressed representations of the images that, for example, can be used to estimate the similarity of two images. In many use cases, though, it is not enough to substitute the images with their feature vectors, but we also need to be able to access the corresponding images. Though the size of the raw image data behind DEEP1B is not stated in [8], an estimate would be 1 MB per image on average, or 1000 TB in total. Clearly, retrieving similar images in a data set of this size is beyond what is possible on a single machine, and even just indexing the set of feature vectors would require an amount of internal memory that is larger than what is present in most servers.

k -nearest neighbors In the k -nearest neighbors (k -NN) problem we want to construct a data structure on a set P of n points in some metric space that, given an integer $k > 0$ and a query point q , finds the k closest points to q in P . We will be focusing on data structures that can be constructed in polynomial time and space. The k -NN problem is believed to be hard in high dimensions even for $k = 1$, and the brute-force algorithm that considers all data points in P essentially optimal. In particular, Williams (see [4]) proved that for constant $\varepsilon > 0$, no $O(n^{1-\varepsilon})$ query time data structure is possible

for $\omega(\log n)$ -dimensional Hamming space, assuming the Strong Exponential Time Hypothesis.

Because of the hardness of the problem most research has revolved around *approximate* solutions. The c -*approximate* k -NN ((c, k) -NN) problem asks to return k points from P with distance at most cr to q , where r is the distance to the k th nearest neighbor of q . It is known that (c, k) -NN is equivalent, up to polylogarithmic factors, to the simpler *near neighbor* problem: Given an upper bound $r_+ \geq r$, return a point within distance cr_+ [16]. We refer to [5] for more background on recent developments in approximate near neighbor search.

Models of computation Motivated by large-scale similarity search applications we consider models of hardware aimed at massive data sets. The *external memory model* [3] abstracts modern, block-oriented storage where memory consists of blocks each capable of holding B data items. The cost of an algorithm or data structure is measured in terms of the number of block accesses, referred to as *I/Os*. When considering the k -NN problem we let B denote the number of vectors that fit in a block.

Distributing similarity search onto many machines has also been considered as a way of scaling to large data sets [10, 29, 20]. We can interpret a static, external memory data structure as an abstraction of a large, distributed system in which each server holds B data items (and information associated with them). In this context the parameter B may be relatively large in comparison to n . For example, to store the $n = 10^9$ DEEP1B vectors of dimension $d = 96$, and associated raw data, we could imagine $s = 10^4$ servers each holding $B = 10^6$ data items (so that on average each item is replicated 10 times to achieve redundancy). The number of I/Os needed to answer a query then equals the number of servers that need to be involved when answering a k -NN query.

Most lower bounds for I/O efficient algorithms are shown under the assumption that data items are *indivisible* in the sense that they are treated as atomic units, and that a block contains a well-defined set of at most B data items. The *indexability model* [17, 18], introduced at PODS '97, formalized external memory data structures for queries that return a set of k data items under the indivisibility assumption. For a given data structure, the complexity of a query family \mathcal{Q} is the smallest number of blocks that must be read to answer a query, in the worst case over all queries $q \in \mathcal{Q}$. There does not

need to be a constructive procedure for *identifying* the correct blocks. In particular, the nearest neighbor problem ($k = 1$) is trivial in the indexability model since the block containing the answer to the query is given for free; the search aspect is completely removed from consideration, and the algorithm would return the block in one I/O. Though the original indexability model does not accommodate notions of approximation, it can be naturally extended to the setting where there is a set of at least k elements that are valid answers to a query and we are required to return any k of them.

1.1 Our results

The complexity of general (c, k) -NN queries in the I/O model lies between two extremes:

- There exist $\lceil k/B \rceil$ blocks (or servers) that contains a set of k valid answers to the query, and
- No block (or server) contains more than one valid query answer, so k block reads are needed.

Since we do not care about constant factors we can, for simplicity, assume that $k < n/2$, since otherwise a trivial brute-force algorithm that reads all points is optimal within a factor of 2. We give several upper and lower bounds for (c, k) -NN that suggest a dichotomy for polynomial space data structures. For various choices of metric space, number of dimensions, and approximation factor we see that it is *either* possible to achieve $O(k/B)$ I/Os, or $\Omega(k)$ I/Os are provably required.

Our results are summarized in Table 1. For our lower bounds, the dimension column represents the minimum (asymptotic) number of dimensions required for the bounds to hold. Since it is possible to decrease the I/O complexity in lower dimensions [35], a condition on the number of dimensions is needed. Our upper bounds do not depend on the number of dimensions, except indirectly through the definition of B as the number of vectors in a block. We stress that the I/O upper bounds for an indexing scheme do not imply the existence of a data structure with the same guarantees — for a data structure in the I/O model we would expect an additional search cost. Our main theorems are:

Theorem 1.1. (*L-infinity metric lower bound*) *Consider an indexing scheme for (c, k) -NN in d -dimensional ℓ_∞ space with $c < 3$ and with worst case query*

| Metric | Dimension | Approximation factor | I/O bound |
|---------------|-----------------------|----------------------|--------------------------|
| any | any | any | $\geq \lceil k/B \rceil$ |
| ℓ_∞ | $\Omega(k \log(n/k))$ | $3 - \varepsilon$ | $\Omega(k)$ |
| Hamming | $\Omega(k \log(n/k))$ | $1 + \frac{1}{4k}$ | $\Omega(k)$ |
| any | any | 3 | $\leq \lceil k/B \rceil$ |
| Hamming | any | $1 + \varepsilon$ | $\leq \lceil k/B \rceil$ |

Table 1: Our I/O lower bounds (first three rows) and upper bounds (last two rows) on the (c, k) -NN problem for data structures using polynomial space, where $\varepsilon > 0$ can be any constant. The lower bound for ℓ_∞ holds assuming $k < n^{1-\varepsilon}$.

time $\lceil k/\alpha \rceil$ I/Os, where $1 \leq \alpha \leq B$. For sufficiently large $d = O(k \log(n/k))$, the indexing scheme must use $\Omega\left(\left(\frac{1}{2e} \sqrt{\frac{n\alpha}{kB}}\right)^\alpha\right)$ blocks of space.

Theorem 1.2. (General metric indexing scheme) Given a set P of n points in any metric space, there exists a 3-approximate indexing scheme that uses $n \lceil k/B \rceil$ blocks of space (where B is the block size) and returns 3-approximate, k nearest neighbors in optimal $\lceil k/B \rceil$ I/Os.

Theorem 1.3. (Hamming metric lower bound) Consider an indexing scheme for (c, k) -NN in d -dimensional Hamming space with $c < 1 + \frac{1}{4k}$ and with worst case query time $\lceil k/\alpha \rceil$ I/Os, where $1 \leq \alpha \leq B$. For sufficiently large $d = O(k \log(n/k))$, the indexing scheme must use $\Omega\left(\left(\frac{1}{2e} \sqrt{\frac{n\alpha}{kB}}\right)^\alpha\right)$ blocks of space.

Theorem 1.4. (Hamming indexing scheme) Given a set P of n points in d -dimensional Hamming space and a constant $c > 1$, there exists a c -approximate indexing scheme that uses $O(n^{O(1)} d \lceil k/B \rceil)$ blocks of space and returns c -approximate, k nearest neighbors in optimal $\lceil k/B \rceil$ I/Os.

Discussion. We are not aware of any studies of c -approximate, k -nearest neighbors in the I/O model. However, our lower bounds suggest that such

data structures cannot simultaneously have good output sensitivity (I/O complexity in terms of k/B) and low space usage. The currently best data structures for high-dimensional, c -approximate r -near neighbor (return 1 point within known distance r) have query time lower bounded by $(n/B)^{f(c)}$ I/Os, for some non-increasing function f with $f(1) = 1$ that depends on the metric. Depending on the choice of c and k this “search cost” may be smaller or larger than the cost of reporting the c -approximate k -nearest neighbors in the indexability model. In general, we would expect the reporting cost to dominate only when c and k are sufficiently large.

Our lower bounds assume that the number of dimensions is sufficiently large. An attempt to bypass the lower bounds would thus be to utilize some kind of *dimension reduction*, such as the Johnson-Lindenstrauss lemma. Unfortunately, this will not work in the settings of Theorems 1.1 and 1.3. This is because dimension reduction is not possible for ℓ_∞ [28], and because dimension reduction for Hamming space with distortion c requires $\Omega(\log(n)/(c-1)^2)$ dimensions [26], which is $\Omega(k^2)$.

1.2 Related work

Lower bounds on nearest neighbors in restricted models A well-known work of Berchtold et al. [12] analyzes the performance of certain types of nearest neighbor indexes on *random* data sets. More recently, Pestov and Stojmirović [33] and Pestov [32] showed lower bounds for high-dimensional similarity search in a model of data structures that encompasses many tree-based indexing methods. These results do not consider approximation, and their algorithmic models do not encompass modern algorithmic approaches to approximate similarity search such as locality-sensitive hashing.

Data structure lower bounds based on indivisibility There is a rich literature, starting with the seminal paper of Aggarwal and Vitter [3], giving lower bounds on I/O-efficiency under an indivisibility assumption. Such results in the context of data structures are known for dynamic dictionaries [37], planar point enclosure [7], range sampling [21], and many variants of orthogonal range reporting [1, 2, 17, 27, 38]. Below we elaborate on the most closely related works on orthogonal range queries.

To our best knowledge the high-dimensional k -NN problem has not been *explicitly* studied in the indexability model [17]. However, in n -dimensional Hamming space it is straightforward to use the *k-set workload* technique

of [17] to show that even obtaining $k - 1$ I/Os is not possible (for $k \ll n$) unless the indexing scheme uses quadratic space. Our lower bound technique is a generalization of the k -set workload that allows us to deal with approximation as well as space usage larger than quadratic. It also allows us to show lower bounds all the way down to $O(k \log n)$ dimensions, as opposed to n dimensions.

Orthogonal range queries *Orthogonal range reporting* in d dimensions asks to report all points in P lying inside a query range \mathcal{Q} that is a cross product of intervals. Note that k -NN in the ℓ_∞ metric is the special case of orthogonal range reporting where all intervals have the same length. Hellerstein et al. [17] showed that in order to answer orthogonal range reporting queries in $O(k/B^{1-\varepsilon})$ I/Os, for some $\varepsilon \in (0, 1)$, the data structure needs to use space $\Omega(n/\varepsilon^{d-1})$. In particular, in dimension $d = \omega(\log n)$ a polynomial space data structure needs $k/B^{o(1)}$ I/Os.

Approximate d -dimensional range reporting has been studied in the I/O model: Streppel and Yi [35] show that for a query rectangle q and constant $\varepsilon > 0$, allowing the data structure to report points at distance up to $\varepsilon \cdot \text{diam}(q)$ from q makes it possible to report k points in $2^{O(d)} + O(k/B)$ I/Os (plus a logarithmic search cost, which does not apply to the indexability model). To our best knowledge, no lower bounds were known for approximate range queries before our work.

Lower bounds based on computational assumptions Unconditional lower bounds for (c, k) -NN in the cell probe model [11, 13, 14, 31] only match upper bounds in the regime where space is very large. To better understand the complexity for, say, sub-quadratic space usage, a possibility is to base lower bounds on computational assumptions such as the Strong Exponential Time Hypothesis (SETH), or the weaker Orthogonal Vectors Hypothesis (OVH). Recently, Rubinfeld [34] showed that under either of these hypotheses, for each constant $\delta > 0$, achieving an approximation factor of $c = 1 + o(1)$ is not possible for a data structure with polynomial space and construction time unless the query time exceeds $n^{1-\delta}$. Already in 2001 Indyk showed that in the ℓ_∞ metric, (c, k) -NN with approximation factor $c < 3$ is similarly hard [22]. (Though Indyk links the lower bound to a different problem, it can be easily checked that the same conclusion follows from the more recent SETH and OVH assumptions.)

Upper bounds The I/O complexity of the near neighbor problem was studied by Gionis et al. [15], focusing on Hamming space. (Their approach extends to other spaces that have good locality-sensitive hash functions.) For every approximation factor $c > 1$, they show that $O((n/B)^{1/c})$ I/Os suffices to retrieve *one* near neighbor, using a data structure of size subquadratic in n . It seems that the same algorithm can be adapted to return k near neighbors at an additional cost of $O(k/B)$ I/Os. Tao et al. [36] extended these results to handle nearest neighbor queries, but they do not consider the case of k nearest neighbors.

For the Euclidean and Hamming metrics with constant approximation factor $c > 1$ it is known how to get $n^{o(1)}$ query time for the near neighbor problem ($k = 1$) with a polynomial space data structure (see [6] and its references). For $c > \sqrt{3}$ Kapralov [24] even showed how to achieve this by a *single* probe to the data structure, returning a pointer to the result.

Organization: The rest of this paper is organized as follows. In Section 2 we develop notation and proceed in Section 2.1 to extend the indexability result of [19] to approximate indexing schemes. Section 3 describes the indexing scheme promised in Theorem 1.2, followed by Section 4 that proves lower bounds in the L-infinity metric (Theorem 1.1). Section 5 contains our results on the Hamming metric (Theorem 1.3 and Theorem 1.4), followed by conclusion and open problems in Section 6.

2 Preliminaries

The external memory model of computation (due to Aggarwal and Vitter [3]) has a main memory of size M and an infinite external memory, both realized as arrays. Data is stored in external memory, and is transferred to/from main memory (where computation happens) in I/Os, or *block transfers*, where a block holds B data items. Computation in main memory is free — the cost of an algorithm is the number of I/Os it performs.

We will be using the following definitions from Hellerstein et al. [17, 18]. For brevity we will refer to a subset of I of size b as a *b-subset*.

Definition 2.1 (Workload). *A workload $W = (D, I, \mathcal{Q})$ consists of a non-empty set D (the domain), a nonempty finite set $I \subseteq D$ (the instance, whose size we denote by n), and a set \mathcal{Q} of subsets of I (the query set).*

Definition 2.2 (Indexing Scheme). *An indexing scheme $S = (W, \mathcal{B})$ consists of a workload $W = (D, I, \mathcal{Q})$ and a set \mathcal{B} of B -subsets of I , where B is the block size of the indexing scheme.*

Definition 2.3 (Cover set). *A cover set $C_Q \subseteq \mathcal{B}$ for a query $Q \in \mathcal{Q}$ is a minimum-size subset of the blocks such that $Q \subseteq \bigcup_{b \in C_Q} b$.*

We will assume that the blocks are chosen such that C_Q exists for every query $Q \in \mathcal{Q}$.

Definition 2.4 (λ -Set Workload). *The λ -set workload is a workload with instance $I = \{1, \dots, n\}$ whose query set \mathcal{Q} is the set of all λ -subsets of the instance.*

While [17, 18] measure performance in terms of *redundancy* and *access overhead*, we find it more natural to define performance in the same way as the I/O model. The *space usage* of an indexing scheme is the number of blocks, $|\mathcal{B}|$. The *query time* of $Q \in \mathcal{Q}$ is the size of a cover set, $|C_Q|$. Observe that the time of a query of size λ can range from $\lceil \lambda/B \rceil$ I/Os to λ I/Os, depending on how many blocks are needed to cover it.

All static data structures we know of in the external memory model with a given space usage and query cost translate directly to an indexing scheme with the same, or better, performance. This is because these data structures store, or can be adapted to store, $O(B)$ vectors explicitly in each block, such that the set of result vectors is explicitly present in the blocks that are read when answering a query. This means that any lower bounds in the indexability model strongly suggest lower bounds also in the external memory model.

2.1 Approximate indexing schemes

Definition 2.5 (c -approximate k -nearest neighbors problem, (c, k) -NN). *Let M be a metric space with distance function $d(\cdot, \cdot)$, and $c > 1$ be a constant. Given $P \subset M$, $|P| = n$, and a positive integer $k \leq n$, construct a data structure that upon receiving a query q , returns $Q \subset P$, $Q = \{q^1, \dots, q^k\}$, such that for all $q^i \in Q$, $d(q, q^i) \leq c d(q, p_q^k)$, where $p_q^k \in P$ is the k th nearest neighbor of q in P .*

We will consider indexing schemes for (c, k) -NN that can depend on the parameters c and k , i.e., the parameters are known when the data structure is constructed. The set of points to be stored is denoted by P . Observe that for (c, k) -NN, there can be at most $\binom{n}{k}$ distinct query answers ($\lambda = k$ in the λ -set workload above). Intuitively, the more queries there are in a workload, the higher is the space needed by the data structure. But in the (c, k) -NN problem, for a query q there is no one “right” answer, as any set of k approximate near neighbors form a valid output. Thus we find that the definitions in the original indexability model need to be extended to capture approximate data structures. It turns out that even though approximate near neighbors are allowed, it is possible to ensure that at least half of the k returned points are among the k nearest to the query point. This motivates the following definition:

Definition 2.6 (Relaxed λ -set workload). *The relaxed λ -set workload is a workload whose instance is $\{1, \dots, n\}$. The query set is the set of all $\binom{n}{\lambda}$ subsets of size λ . Given a query q corresponding to such a λ -subset Q , the indexing scheme must report λ elements, at least half of which must come from Q .*

We next show a space-I/O tradeoff for the relaxed workload.

Lemma 2.1 (Relaxed set workload tradeoff). *Any indexing scheme for the relaxed λ set workload with a space usage of s blocks and a query time of t block accesses must have*

$$t \geq \lambda \log_s \left(\frac{1}{2e} \sqrt{\frac{n}{tB}} \right)$$

Proof. Recall that by the relaxation, the indexing only needs to find a subset $\hat{Q} \subset Q$ of size $|\hat{Q}| = \lambda/2$ in the index. In other words, with each subset \hat{Q} that can be retrieved by the index, the algorithm can add any $\lambda/2$ elements and arrive at a valid query Q .

We upper bound the total number of distinct λ -sets reported by an indexing scheme using s space and t query I/Os as follows:

- choose the set of t blocks to retrieve from the index ($\binom{s}{t}$ choices),
- choose the $\lambda/2$ elements to use from these at most tB distinct elements (at most $\binom{tB}{\lambda/2}$ choices),

- choose $\lambda/2$ arbitrary *other* elements (at most $\binom{n}{\lambda/2}$ choices).

The total number of such combinations should be at least $\binom{n}{\lambda}$, which is the possible set of queries, or λ -subsets. Using the inequalities $\binom{n}{k}^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k \leq n^k$ this gives us:

$$\begin{aligned}
\binom{s}{t} \binom{tB}{\lambda/2} \binom{n}{\lambda/2} &\geq \binom{n}{\lambda} & (1) \\
\Rightarrow s^t \left(\frac{tBe}{\lambda/2}\right)^{\lambda/2} \left(\frac{ne}{\lambda/2}\right)^{\lambda/2} &\geq \left(\frac{n}{\lambda}\right)^\lambda \\
\Rightarrow s^{2t/\lambda} \left(\frac{tBe}{\lambda/2}\right) \left(\frac{ne}{\lambda/2}\right) &\geq \left(\frac{n}{\lambda}\right)^2 \\
\Rightarrow s^{2t/\lambda} &\geq \frac{n}{4e^2tB} \\
\Rightarrow t &\geq \lambda \log_s \left(\frac{1}{2e} \sqrt{\frac{n}{tB}}\right).
\end{aligned}$$

□

One may be interested in a query time of $\lceil \lambda/\alpha \rceil$ I/Os, where $1 \leq \alpha \leq B$.

Corollary 2.2. *Any indexing scheme for the relaxed λ -set workload with worst case query time of $\lceil \lambda/\alpha \rceil$ I/Os, where $1 \leq \alpha \leq B$ must use $\Omega\left(\left(\frac{1}{2e} \sqrt{\frac{n\alpha}{\lambda B}}\right)^\alpha\right)$ blocks of space.*

This lower bound is essentially tight: Achieving a query time of $\lceil \lambda/\alpha \rceil$ I/Os using $O\left(\binom{n}{\alpha}\right)$ space is easy by just preprocessing all α combinations of the n items. By doing an analogous calculation for the exact workload, we are able to give the following tradeoff for the standard λ -set workload.

Lemma 2.3 (λ -set workload tradeoff). *Any indexing scheme for the λ -set workload with worst case query time of $\lceil \lambda/\alpha \rceil$ I/Os, where $1 \leq \alpha \leq B$ must use $\Omega\left(\left(\frac{n\alpha}{e\lambda B}\right)^\alpha\right)$ blocks of space.*

Lemma 2.3 generalizes Theorem 7.2 in [17] which considered the case of $\alpha = 1$.

3 A 3-approximation indexing scheme for general metrics

We prove Theorem 1.2 in this section, that asserts that (a relatively simple) indexing scheme provides a 3-approximation for the k -NN problem in any metric space. Note that we are only presenting an indexing scheme as opposed to a data structure; i.e., we assume that once the query is given, an oracle provides the smallest set of blocks that contain a valid answer to the query.

Let P be a set of n points in a metric space. Consider the indexing scheme I that consists of the set of $\ell = n \lceil k/B \rceil$ blocks $\mathcal{B} = \{b_1, b_2, \dots, b_\ell\}$ where we store $p_i \in P$ and the set $P_i = \{p_{i,1}, p_{i,2}, \dots, p_{i,k}\}$ of the k -nearest neighbors of the point p_i (including itself). This requires $\lceil k/B \rceil$ blocks per element of P , as claimed. For a query point q let p_{i^*} be the nearest neighbor of q . The oracle then returns the set P_{i^*} , i.e., the k nearest neighbors of q 's nearest neighbor, using $\lceil k/B \rceil$ I/Os.

Theorem 3.1. *Let $\{p_1^q, \dots, p_k^q\}$ be the set of the exact k nearest neighbors of a query q , and let $\{p_{i^*,1}, \dots, p_{i^*,k}\}$ be the set of points returned by the indexing scheme described above. Then for any $1 \leq j \leq k$, $d(q, p_{i^*,j}) \leq 3d(q, p_k^q)$. That is, all the returned points are within a factor 3 of the distance of the query to its k th nearest neighbor.*

Proof. The proof is a case analysis. Let D be the smallest ball centered at $p_1^q := p_{i^*}$ that contains $p_{i^*,k}$ (the “ k nearest neighbor disk” of p_{i^*}). The cases are:

1. Both q and p_k^q are outside D . For each j , we have that

$$\begin{aligned}
 d(q, p_{i^*,j}) &\leq d(p_{i^*,j}, p_1^q) + d(p_1^q, q) \quad (\text{triangle inequality}) \\
 &\leq d(p_{i^*,j}, p_1^q) + d(q, p_k^q) \quad (p_k^q \text{ is farther from } q \text{ than } p_1^q) \\
 &\leq d(q, p_1^q) + d(q, p_k^q) \quad (q \text{ is outside } D) \\
 &\leq 2d(q, p_k^q) .
 \end{aligned}$$

2. q is inside D , and p_k^q is outside D . For each j , we have that

$$\begin{aligned}
d(q, p_{i^*,j}^q) &\leq d(p_{i^*,j}^q, p_1^q) + d(p_1^q, q) \quad (\text{triangle inequality}) \\
&\leq d(p_{i^*,j}^q, p_1^q) + d(q, p_k^q) \quad (p_k^q \text{ is farther from } q \text{ than } p_1^q) \\
&\leq d(p_k^q, p_1^q) + d(q, p_k^q) \quad (p_k^q \text{ is outside } D) \\
&\leq d(p_k^q, q) + d(q, p_1^q) + d(q, p_k^q) \quad (\text{triangle inequality}) \\
&\leq 3 d(q, p_k^q) .
\end{aligned}$$

3. Both q and p_k^q are inside D . In this case, we first claim that it is sufficient to consider that for some $i \notin \{1, k\}$, p_i^q is outside D . If that was not the case, we have reported all $\{p_i^q\}$, i.e., the exact k nearest neighbors of q . For each j we now have that

$$\begin{aligned}
d(q, p_{i^*,j}^q) &\leq d(q, p_1^q) + d(p_1^q, p_{i^*,j}^q) \quad (\text{triangle inequality}) \\
&\leq d(q, p_k^q) + d(p_1^q, p_{i^*,j}^q) \quad (p_k^q \text{ is farther from } q \text{ than } p_1^q) \\
&\leq d(q, p_k^q) + d(p_1^q, p_i^q) \quad (p_i^q \text{ is outside } D) \\
&\leq d(q, p_k^q) + d(p_1^q, q) + d(q, p_i^q) \quad (\text{triangle inequality}) \\
&\leq 3 d(q, p_k^q) .
\end{aligned}$$

4. q is outside D , and p_k^q is inside D . In this case we have that

$$\begin{aligned}
d(q, p_{i^*,j}^q) &\leq d(p_{i^*,j}^q, p_1^q) + d(p_1^q, q) \quad (\text{triangle inequality}) \\
&\leq d(q, p_1^q) + d(p_1^q, q) \quad (q \text{ is outside } D; p_{i^*,j}^q \text{ is inside}) \\
&\leq 2d(q, p_k^q) \quad (p_k^q \text{ is farther from } q \text{ than } p_1^q)
\end{aligned}$$

□

Tightness of the analysis: Next, we give an example of $n = k + 1$ points where the indexing scheme above does not achieve a better approximation factor than $3(1 - \varepsilon)$, for arbitrary $\varepsilon > 0$. Consider the scenario when all points including the query lie on a line:

$$\begin{aligned}
q = 1/2, \quad p_1^q = 0, \quad p_k^q = 1 + \varepsilon/2, \quad p_{k+1}^q = -1, \quad \text{and} \\
p_i^q = -\varepsilon i/2k \text{ for all } i \notin \{1, k, k + 1\} .
\end{aligned}$$

The disk D contains all points but p_k^q, p_{k+1}^q will be reported, and $d(q, p_{k+1}^q) = 3/2 > 3(1 - \varepsilon)(1/2 + \varepsilon/2) = 3(1 - \varepsilon)d(q, p_k^q)$.

4 Lower bound for L-infinity metric

In this section we prove Theorems 1.1 and 1.3.

4.1 Lower bound for the L-infinity metric

We warm up with a proof of a lower bound in n dimensions, a slight variant of a reduction by Indyk [22].

Lemma 4.1. *Consider the approximate k -NN problem in \mathbb{R}^n with ℓ_∞ metric. There exists a set of n points p_i for $i \in \{1, \dots, n\}$ that has a k -set workload. Specifically, for every subset $I \subset [n]$ there exists a query point q_I such that the ℓ_∞ -distance is*

$$\|p_i - q_I\|_\infty = \begin{cases} 1/2 & \text{for } i \in I \\ 3/2 & \text{for } i \notin I \end{cases} .$$

Hence, the (c, k) -NN for $c < 3$ in n -dimensional ℓ_∞ space leads to a k -set workload.

Proof. The set consists of the n unit vectors $p_i = e_i$ (where only the i th entry is 1 and all other entries are 0). For a set $I \subset \{1, \dots, n\}$ the query vector is defined as

$$q_I = \begin{cases} +1/2 & \text{for } i \in I \\ -1/2 & \text{for } i \notin I \end{cases} .$$

□

To deterministically reduce the dimensionality of the space we use an expander and switch to relaxed k -set workloads. Expanders were previously used for deterministic embeddings of Euclidean space into ℓ_1 by Indyk [23]. There is a vast literature on expanders and the results we are using are standard by now. For the sake of concreteness, we take the definitions and precise results almost literally from [30]. We define $(m, \delta, 1/3)$ -expander graphs and state some results concerning these graphs. For the rest of this paper we will assume δ to divide $1/3$, as this makes statements and proofs simpler. This will be without loss of generality, as the statements we show do not change when rounding δ down to the nearest such value. Let $G = (U, V, E)$ be a bipartite graph with left vertex set U , right vertex set V , and edge set E . We denote the set of neighbors of a set $S \subseteq U$ by $\Gamma(S) = \cup_{s \in S} \{v \mid (s, v) \in E\}$, and use $\Gamma(x)$ as a shorthand for $\Gamma(\{x\})$, $x \in U$.

Definition 4.1 (Definition 3 of [30]). A bipartite graph $G = (U, V, E)$ is δ -regular if the degree of all nodes in U is δ . A bipartite δ -regular graph $G = (U, V, E)$ is an $(m, \delta, 1/3)$ -expander if for each $S \subseteq U$ with $|S| \leq m$ it holds that $|\Gamma(S)| \geq (1 - 1/3)\delta|S|$.

Lemma 4.2 (Corollary 5 in [30]). For every constant $\varepsilon > 0$ there exists an (m, δ, ε) -expander $G = (U, V, E)$ with $|U| = u$, $\delta = O(\log(2u/m))$ and $|V| = O(m \log(2u/m))$.

Next, we discuss how to give an analogue of the hard query set in Lemma 4.1 with $O(k \log n)$ dimensions.

Lemma 4.3. Let $n, k(n)$ be arbitrary integer parameters. Consider (c, k) -NN with $c < 3$. There exists a set D of n points in dimension $O(k \log n)$ such that for any $I \subset [n]$ with $|I| = k$ there exists a query point q_I such that the set $I' = \{p \in D \mid \|p - q_I\|_\infty \leq 1/2\}$ of potential answer points has $I \subset I'$ and $|I'| \leq 3k/2$.

Proof. Fix n and $k(n)$. Let $m = k(n)$ and let $G = (U, V, E)$ be an $(m, \delta = O(\log(n/m)), 1/3)$ -expander with $|U| = n$ and $|V| = d = O(k\delta)$. For concreteness we take $U = [n]$.

Construct the set of points p_1, \dots, p_n where

$$(p_i)_j = \begin{cases} 1 & \text{if } j \in \Gamma(i) \\ 0 & \text{otherwise} \end{cases} .$$

Define the query point for set I with $|I| = k$ where

$$(p_I)_j = \begin{cases} +1/2 & \text{for } j \in \Gamma(I) \\ -1/2 & \text{otherwise} \end{cases} .$$

It is easy to see that $\|p_i - p_I\|_\infty = 1/2$ for all $i \in I$, so p_I has at least k neighbors at distance $1/2$. It remains to show that this leads to a relaxed k -set workload, i.e., that for any q_I any set of $1/2$ -near points in the set has at least $k/2$ points in common with I . Fix a subset I and consider the query point q_I . Let $I' = \{i \mid \|p_i - p_I\|_\infty \leq 1/2\}$ be the indices of the points of with distance at most $1/2$ to q_I . Observe that $I \subseteq I'$ by construction. Let $I^* = I' \setminus I$ be the indices of “unintended near points”. Observe that every point p_i that does not fulfill $\Gamma(i) \subseteq \Gamma(I)$ has $\|p_i - p_I\|_\infty = 3/2$ and cannot be reported since $c < 3$. Hence

$$I^* = \{i \mid \Gamma(i) \subseteq \Gamma(I)\} \setminus I .$$

Observe $|\Gamma(I)| \leq \delta|I|$ and $|\Gamma(I \cup I^*)| \geq (1 - 1/3)\delta|I \cup I^*|$, by the definition of G and $|\Gamma(I \cup I^*)| = |\Gamma(I)|$ by the definition of I^* . Combining this we get $\delta|I| \geq \frac{2}{3}\delta|I \cup I^*|$, or $|I| \geq \frac{2}{3}(|I| + |I^*|)$. That is, $\frac{1/3}{2/3}|I| \geq |I^*|$, as desired. Hence we have a relaxed k -set workload. \square

This means that we can apply Lemma 2.1 and Corollary 2.2:

Corollary 4.4. *(Theorem 1.1) Any indexing scheme for (c, k) -NN in $O(k \log(n/k))$ -dimensional ℓ_∞ space with $c < 3$ with worst case query time of $\lceil k/\alpha \rceil$ I/Os, where $1 \leq \alpha \leq B$ must use $\Omega\left(\left(\frac{1}{2e}\sqrt{\frac{n\alpha}{kB}}\right)^\alpha\right)$ blocks of space.*

4.2 Lower Bound for the Hamming metric

We now prove Theorem 1.3, giving a lower bound on indexes on sets of vectors in Hamming space with approximation factor $c > 1$. This directly implies a lower bound in the ℓ_1 metric, as well as a lower bound for ℓ_2 with approximation factor \sqrt{c} .

Lemma 4.5. *Consider the approximate k -NN problem in Hamming space of dimension d . There exists a set of n points p_i , $i \in \{1, \dots, n\}$ of dimension $d = n$ that is a k -set workload, i.e, for every subset $I \subset [n]$ with $|I| = k$ there exists a query point q_I such that the Hamming (or ℓ_1) distance is:*

$$\|p_i - q_I\|_1 = \begin{cases} k - 1 & \text{for } i \in I \\ k + 1 & \text{for } i \notin I \end{cases} .$$

Hence, the (c, k) -NN for $c < 1 + 2/(k - 1)$ in n -dimensional Hamming space leads to a k -set workload.

Proof. The set consists of the n unit vectors e_i . The query vector for set I is the characteristic vector of I . It is easy to verify that the distances from the query to the vectors in I is $k - 1$, and to those not in I is $k + 1$. Note that $(k + 1)/(k - 1) = 1 + 2/(k - 1)$, which gives the bound on c . \square

Lemma 4.6. *Consider the approximate k -NN problem in Hamming space of dimension d . There exists a set of n points p_i , $i \in \{1, \dots, n\}$, and $d = O(k \log(n/k))$ that is a relaxed k -set workload, i.e, for every subset $I \subset [n]$ with $|I| = k$ there exists a query point q_I such that the Hamming (ℓ_1)-distance is*

$$\|p_i - q_I\|_1 = \begin{cases} \leq d_I & \text{for } i \in I \\ \geq d_I \left(1 + \frac{1}{4(k-1)}\right) & \text{for } i \notin I \end{cases}$$

for an $I' \supset I$ with $|I'| \leq 1.5|I|$. Hence, $c < 1 + \frac{1}{4^{(k-1)}}$ leads to a relaxed k -set workload for (c, k) -NN in $O(k \log n)$ -dimensional Hamming space.

Proof. Fix n and $k(n)$. Let $G = (U, V, E)$ be an $(m = k(n), \delta = O(\log n), 1/4)$ -expander with $|U| = n$ and $|V| = d = O(k \log n)$, which exists by Lemma 4.2. For concreteness we take $U = [n]$.

Construct the set of points p_1, \dots, p_n in the same way as earlier:

$$(p_i)_h = \begin{cases} 1 & \text{for } h \in \Gamma(i) \\ 0 & \text{otherwise} \end{cases} .$$

We define the query point q_I for each set I with $|I| = k$ where

$$(q_I)_h = \begin{cases} 1 & \text{for } h \in \Gamma(I) \\ 0 & \text{otherwise} \end{cases} .$$

In other words, the vectors are the characteristic vectors of the neighbor-sets in G .

We define $d_j := \|p_j - q_I\|_1$. For $i \in I$ we have $d_i = |\Gamma(I)| - \delta$ and

$$d_i \geq k\delta(1 - 1/4) - \delta = \delta(k(1 - 1/4) - 1) .$$

For any j (in particular $j \notin I$) we have $d_j = \|p_j - q_I\|_1 = |\Gamma(I)| - \delta + 2|\Gamma(j) \setminus \Gamma(I)|$, leading to the definition $\gamma_j = |\Gamma(j) \setminus \Gamma(I)|$. We can set the distance threshold ratio such that the number of unintended neighbors is again at most $k/2$:

We can set the distance threshold ratio to $1 + 1/(4(k-1))$ such that the set of unintended near neighbors are the j with $\gamma_j < \delta/4$:

$$d_j/d_I = 1 + \gamma_j/d_I \leq 1 + 1/(4(k-1)) \Rightarrow \gamma_j \leq d_I/(4(k-1)) \leq \delta/4$$

To calculate the set of indices of unintended neighbors we define

$$I^* = \{j \mid \gamma_j \leq \delta/4\} \setminus I .$$

Then

$$\begin{aligned} |\Gamma(I \cup I^*)| &\leq \delta|I| + (\delta/4)|I^*| \\ |\Gamma(I \cup I^*)| &\geq (1 - 1/4)\delta|I \cup I^*| = \frac{3}{4}\delta(|I| + |I^*|) \end{aligned}$$

leading to

$$\delta|I| + (\delta/4)|I^*| \geq \frac{3}{4}\delta(|I| + |I^*|)$$

$$\begin{aligned} &\Rightarrow (\delta/4)|I^*| \geq \frac{3}{4}\delta|I^*| - |I|\delta/4 \\ &\Rightarrow |I|\delta/4 \geq \frac{3}{4}\delta|I^*| - (\delta/4)|I^*| . \end{aligned}$$

Hence

$$|I^*| \leq |I| \frac{1/4}{\frac{3}{4} - 1/4} = k/2 .$$

This means that the described workload is a relaxed k -set workload. Applying Lemma 2.1 and Corollary 2.2 now completes the proof of Theorem 1.3. \square

5 Hamming metric indexing scheme

The general 3-approximate indexing scheme described in Section 3 can be improved for specific metrics. In this section we prove Theorem 1.4. For any given approximation factor $c > 1$ we wish to construct an indexing scheme that answers (c, k) -NN queries in d -dimensional Hamming space using polynomial space and with $\lceil k/B \rceil$ I/Os. Our construction is an application of the dimension reduction technique of Kushilevitz et al. [25].

For each $r \in \{1, \dots, d\}$ we create a data structure that handles the case where the k th closest point to q is at distance r . The data structure must report k points that have distance at most cr from q . The central idea of [25] is to use a randomized mapping

$$t : \{0, 1\}^d \rightarrow \{0, 1\}^D,$$

where $D = O(\log n)$, such that for each $q \in \{0, 1\}^d$ with high probability for all $x, y \in P$:

$$d(q, x) \leq r \wedge d(q, y) > cr \implies d(t(q), t(x)) < d(t(q), t(y)) . \quad (2)$$

(We note that the required dimension D grows as c approaches 1, hence we need to keep c fixed.) Consider the mapped multiset $t(P) = \{t(x) \mid x \in P\}$ and create a data structure that for each $i \in \{0, 1\}^D$ lists, for the k nearest neighbors of i in $t(P)$, the corresponding vectors in P (breaking ties arbitrarily), using $\lceil k/B \rceil$ blocks. If (2) holds then list $i = t(q)$ contains only c -approximate k -nearest neighbors of q . To eliminate the error probability, choose $O(d)$ such random mappings and construct corresponding data structures: With high probability there will be no query $q \in \{0, 1\}^d$ that does not

have at least one data structure that returns a correct result. If this fails for some q , start over from the beginning and choose new mappings.

The total space usage is $O(2^D d \lceil k/B \rceil)$, which is polynomial in n and d , as desired. Queries can be answered in $\lceil k/B \rceil$ I/Os since we are taking full advantage of the power of the indexability model: To answer a query q it is necessary to know which mapping t can be used to answer it correctly, and where in storage the blocks with index $t(q)$ reside.

Of course, we can also get an algorithm in the standard I/O model with a multiplicative query time overhead of $O(d)$ by querying *all* repetitions and returning the k closest points seen.

6 Conclusion and open problems

We have shown that nontrivial lower bounds can be shown in the indexability model, even under approximation. The main open problem that we leave is whether our hardness result for Hamming distance can be extended to approximation factor $c = 1 + \Omega(1)$, where the constant in $\Omega(1)$ is independent of k . This would give an unconditional analogue of the recent conditional lower bound of Rubinfeld [34].

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